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SOME USEFUL INFORMATION FOR THE DESIGN OF AIR-CORE SOLENOIDS

by

D. Bruce Montgomery
J. Terrell

November, 1961
(2nd Printing, December 1961)

Sponsored by the Solid State Sciences Division,
Air Force Office of Scientific Research (OAR).

Air Force Contract AF 19(604)-7344

Principal Investigators: Benjamin Lax and Francis Bitter

National Magnet Laboratory
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

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PART I: Relationships between Magnetic Field, Power, Ampere-Turns and Current Density

PART II: Homogenous Magnetic Fields

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THE AUTHORS ARE INDEBTED TO F. BITTER FOR HIS MANY CONTRIBUTIONS TO THIS PAPER.

ABSTRACT

Relationships relating power, magnetic field, current density, and ampere-turns in terms of certain dimensionless factors are summarized for many types of coil geometries and current distributions. A number of plots of these factors are presented.

The field homogeneity in magnet structures is presented in terms of a series expansion about the origin utilizing Legendre Polynomials. A number of tables to facilitate design of homogeneous fields are presented. A method of achieving homogeneity in long solenoid structures by the use of determinants is discussed. Expressions for the axial field from uniform and radially varying current density coils are given.

PART I

RELATIONSHIPS BETWEEN MAGNETIC FIELD, POWER, AMPERE-TURNS AND CURRENT DENSITY

It has been customary in the literature on magnet design to write the relationships between fields, power, ampere-turns and current density in terms of a number of dimensionless factors. In this section we will treat in summary form twelve cases of coil construction (Cases I - XII). This work is in part a summary, and enlargement of parts of the following three papers: F. Bitter, R.S.I. Vol. 7, 1936 Part II; W. F. Gauster, AIEE, Fall General Meeting, Chicago, October 1959; F. Gaume, Journal de Recherche du Centre Nationale de Recherche Scientifique, No. 43, June 1958. (English translation, Lincoln Laboratory, M81-12, part I and II, Dr. H. H. Kolm.)

A. The most commonly used dimensionless factor is the G factor first suggested by Fabry, which connects a given magnetic field with the power required to produce this field and is a function of the type of current distribution and the geometry of the coil.

$$H = G\left(\frac{W\lambda}{\rho a_1}\right)^{1/2} \qquad (1)$$

with W = watts

ρ = resistivity in ohm - cm

a_1 = inner radius in cm

λ = fractional volume of conductor

The space factor λ is assumed constant throughout the volume. If λ is a function of x and y, more complicated formulas are needed.

B. A second dimensionless factor J has been suggested by W. F. Gauster and connects the current density at the innermost winding of a magnet to the power of the magnet. J is also a function of the current distribution and the geometry of the coil.

$$j_1 = J\left(\frac{W}{\rho\lambda a_1^3}\right)^{1/2} \qquad (2)$$

C. For any particular type of coil a maximum G factor can be found which will give the most magnetic field for the least power used. However, in many types of coil construction this maximizing of the G factor may lead to excessively high current densities at the innermost winding which may make cooling the coil difficult. To illustrate this point a ratio, γ , can be formed which compares the maximum current density in any coil to the current density in a uniform current density coil, where both have the same inner radius and both develop the same field. Quantities for the uniform case are designated as primed:

$$H = G\left(\frac{W\lambda}{\rho a_1}\right)^{1/2} \quad H' = G'\left(\frac{W'\lambda}{\rho a_1}\right)^{1/2}$$

if $H = H'$

$$\left(\frac{W}{W'}\right)^{1/2} = \frac{G'}{G} \tag{3}$$

from (2)

$$\frac{j_1}{j_1'} = \frac{J}{J'}\left(\frac{W}{W'}\right)^{1/2}$$

and using (3)

$$\frac{j_1}{j_1'} = \frac{J}{J'}\frac{G'}{G} = \gamma \tag{4}$$

This ratio indicates the difficulty of cooling and supporting any coil compared with a coil of uniform current density at the same field and bore.

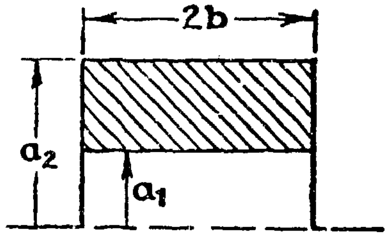
D. The ratio in equation (3) is also useful and represents the power saving in using any current distribution compared with uniform current density when generating the same field in the same bore. We give it in the more useful form.

$$\frac{W}{W'} = \left(\frac{G'}{G}\right)^2 = P \tag{5}$$

These dimensionless factors and ratios are presented in Table I for nine types of magnet construction and following Table I for three additional more complicated magnet constructions.

One of the most interesting new coil constructions presented is that of case V. This coil was first suggested by F. Gaume and achieves a relatively high G factor without excessive current densities and can be quite simply constructed of plates in the usual Bitter magnet style, with the introduction of axial current variations by making the plates thicker towards the ends of the magnet.

A number of plots of the geometry factor, G, against α and β are given in Figures 1 through 6. Cockcroft's curves for case VIII are reprinted in Figure 1, Bitter's curves for case VI in Figure 2, new curves for cases VIII and VI in the small α and β region are given in Figures 3 and 4. New curves for case V (Gaume's axial current variation) and for case X (the ellipsoidal shell) are given in Figures 5 and 6. The variables α and β are defined:



$$\alpha = a_2/a_1$$
$$\beta = b/a_1$$

E. It is sometimes instructive to write the magnetic field in terms of ampere-turns rather than power. This arises from the practical standpoint that it is difficult to always predict the resistance of a magnet exactly, especially as a function of power, and one may wish to know how much field will be produced for each ampere available in the power supply. This relationship between ampere-turns and field can be related to the J factor and the G factor and miscellaneous α and β terms and numerical constants. The relationships are given here for three cases: Case VI, Case VII and Case VIII. All the others are easily derivable by integrating the current density over the coil to obtain the total current in terms of J and W. Total current I and the factor J can now be substituted for W in equation (1), and the ampere-turns related to the field.

Case VI:

$$H_6 = \frac{I\lambda}{t}\left[\frac{1}{J_6 \ln\alpha}\right]G_6 \tag{6}$$

$$= \frac{NI}{2b}\left[\frac{1}{J_6 \ln\alpha}\right]G_6$$

Case VII:

$$H_7 = \frac{I\lambda}{t}\left[\frac{1}{J_7}\right]G_6 \tag{7}$$

$$= \frac{NI}{ka_1}\left[\frac{1}{(\alpha-1)J_7}\right]G_7$$

Case VIII:

$$H_8 = \frac{I\lambda}{t}\left[\frac{1}{J_8(\alpha-1)}\right]G_8 \tag{8}$$

$$= \frac{NI}{2b}\left[\frac{1}{J_8(\alpha-1)}\right]G_8$$

F. Useful geometry factors can be written for very long solenoids (long enough so that end effects can be neglected).

For a constant current density coil the ampere-turns per unit length can be written:

$$\frac{NI}{2b} = \left(\frac{(W/2b)\lambda}{2\pi\rho}\right)^{1/2}\left(\frac{2(\alpha-1)}{\alpha+1}\right)^{1/2} \tag{9}$$

where W/2b is the power per unit length (watts/centimeter)

For the disk construction, with a radial current distribution we write

$$\frac{NI}{2b} = \left(\frac{(W/2b)\lambda}{2\pi\rho}\right)^{1/2}\left(\ln \alpha\right)^{1/2} \tag{10}$$

Rewriting (9) and (10) in terms of the field in gauss and power per length as watts per meter, we obtain

$$H(\text{gauss}) = \frac{(2\pi)^{1/2}}{50}\, G_5 \left(\frac{W\,(\text{watts/meter})\,\lambda}{\rho\,(\text{ohm cm.})}\right)^{1/2}$$

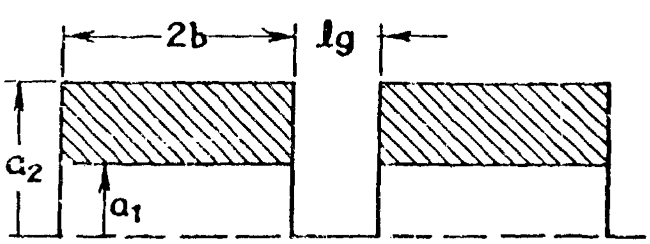
where

$$G_5\,(\text{uniform}) = \left(\frac{2(\alpha-1)}{\alpha+1}\right)^{1/2}$$

$$G_5\,(\text{radial}) = (\ln \alpha)^{1/2}$$

These factors are plotted in Figure 7 and it is interesting to note that below an α of 2.5 there is negligible power saving with the disks, but at greater α, the saving is increasingly pronounced; at an α of 10, 45% more power would be required by the uniform wound solenoid for the same magnetic field.

G. It is useful to use the G factor notation when dealing with split coils (coils separated by a gap). If we take a pair of coils with parameters as defined in Figure 8, we can write the field in the gap by simple addition and subtraction of coils.



$$\alpha = a_2/a_1$$

$$\beta_1 = \frac{\text{total length}}{2a_1} = \frac{4b+\ell g}{2a_1}$$

$$\beta_2 = \frac{\text{gap}}{2a_1} = \frac{\ell g}{2a_1}$$

Figure 8

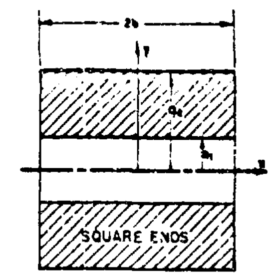
$$H(0,0) = \left(\frac{W\lambda}{\rho a_1}\right)^{1/2} \left(\frac{G_1 \beta_1^{1/2} - G_2 \beta_2^{1/2}}{(\beta_1 - \beta_2)^{1/2}}\right)$$

$$G_1 = G_1(\alpha, \beta_1), \quad G_2 = G_2(\alpha, \beta_2)$$

This relationship (11) holds for any current distribution having no axial variation.

TABLE 1

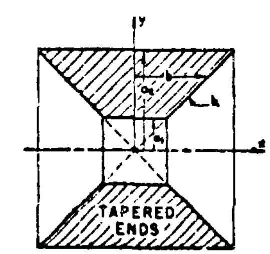
	Shape of Coil	Current Distribution	G Geometry Factor	$G_{max.}$	J Current Density Factor	$J_{max.}$	γ Current Density Ratio for Same Field Referred to Case VIII	P Power Ratio for Same Field Referred to Case VIII
I	Spherical Shell	Optimum Current Distribution $i=f(x,y)\sim\frac{y}{x(x^2+y^2)}$	$\frac{1}{5}\left(\frac{2\pi}{3}\right)^{1/2}\left(\frac{\alpha-1}{\alpha}\right)^{1/2}$	α = INF. 0.289	$\left(\frac{\alpha}{2\pi(\alpha-1)}\right)^{1/2}$	$\alpha=10$ $G=.275$ 0.421	2.74	0.421
II	Cylindrical Bore Square Ends (Maxwell Coil)	Optimum Current Distribution $i=f(x,y)\sim\frac{y}{(x^2+y^2)^{3/2}}$	$\frac{(2\pi)^{1/2}}{10}\left(\frac{1}{4}\left(\frac{1}{1+\beta^2}-\frac{\beta}{\alpha^2+\beta^2}\right)-\frac{1}{4\alpha}\tan^{-1}\frac{\beta}{\alpha}+\frac{3}{4}\tan^{-1}\beta\right)^{1/2}$	α,β = INF. 0.272	$\frac{1}{10G}$	$\alpha=\beta=10$ $G=.255$ 0.392	2.75	0.495
III	Cylindrical Bore Tapered End	Optimum Current Distribution $i\sim\frac{1}{y^2}$ For Tapered Ends	$\frac{1}{5}\left(\frac{\pi k}{k^2+1}\right)^{1/2}\left(\frac{\alpha-1}{\alpha}\right)^{1/2}$ (Max when $k=1$)	$k=1$, α = INF. 0.250	$\left(\frac{\alpha}{(\alpha-1)4\pi k}\right)^{1/2}$	$k=1$, $\alpha=10$ $G=.138$ 0.298	2.24	0.568
IV	Cylindrical Bore Square Ends	Optimum Radial Current Distribution $i=f(y)\sim\frac{1}{y(b^2+y^2)^{1/2}}$	$\frac{1}{10}\left(\frac{2\pi}{\beta}\ln\frac{\alpha^2(1+\beta^2)}{\alpha+\beta}\right)^{1/2}$	α = INF., $\beta=2$ 0.225	$\frac{1}{10G}$	$\alpha=10$, $\beta=3$ $G=.216$ 0.463	3.85	0.651
V FIGURE 5	Cylindrical Bore Square Ends (Gaume Coil)	$i\sim\frac{1}{y}\left[\frac{1}{(x^2+a_1^2)^{1/2}}-\frac{1}{(x^2+a_2^2)^{1/2}}\right]\frac{a_1a_2}{a_2-a_1}$	$\frac{(\pi)^{1/2}}{5}\left(\frac{1}{\ln\alpha}\right)^{1/2}\left(\tan^{-1}\beta+\frac{1}{\alpha}\tan^{-1}\frac{\beta}{\alpha}-\frac{2}{\alpha}F(\phi,k)\right)^{1/2}$ $F(\phi,k)$ = Elliptic of 1st kind $\phi=\tan^{-1}\beta$, $k^2=\frac{\alpha^2-1}{\alpha^2}$	$\alpha=\beta=6$ 0.232	$\frac{\alpha-1}{10G\alpha\ln\alpha}$	$\alpha=\beta=6$ $G=.232$ 0.181	1.40	0.592
VI FIGURES 2,4	Cylindrical Bore Square Ends	$i\sim\frac{1}{y}$	$\frac{(\pi)^{1/2}}{5}\left(\frac{1}{\beta\ln\alpha}\right)^{1/2}\ln\alpha\frac{\beta\alpha(1+\beta^2)^{1/2}}{\beta+(\alpha^2+\beta^2)^{1/2}}$	$\alpha=6$, $\beta=2$ 0.209	$\left(\frac{1}{4\pi\beta\ln\alpha}\right)^{1/2}$	$\alpha=6$, $\beta=2$ $G=.209$ 0.149	1.30	0.726
VII	Cylindrical Bore Tapered Ends	$i\sim\frac{1}{y}$	$\frac{1}{5}\left(\frac{\pi k}{k^2+1}\right)^{1/2}\frac{\ln\alpha}{(\alpha^2-1)^{1/2}}$ (Max when $K=1$)	$k=1$, $\alpha=4.5$ 0.201	$\left(\frac{1}{4\pi k(\alpha-1)}\right)^{1/2}$	$k=1$, $\alpha=4.5$ $G=.201$ 0.151	1.56	0.796
VIII FIGURES 1,3	Cylindrical Bore Square Ends	i = Constant	$\left(\frac{2\pi\beta}{\alpha^2-1}\right)^{1/2}\frac{1}{5}\ln\frac{\alpha+(\beta^2+\alpha^2)^{1/2}}{1+(\beta^2+1)^{1/2}}$	$\alpha=3$, $\beta=2$ 0.179	$\left(\frac{1}{2\pi\beta(\alpha^2-1)}\right)^{1/2}$	$\alpha=3$, $\beta=2$ $G=.179$ 0.099	1.00	1.000
IX	Cylindrical Bore Tapered Ends	i = Constant	$\frac{1}{5}\left(\frac{3\pi k}{k^2+1}\right)^{1/2}\frac{\alpha-1}{(\alpha^3-1)^{1/2}}$ (Max when $K=1$)	$k=1$, $\alpha=2.7$ 0.172	$\left(\frac{3}{4\pi k(\alpha^3-1)}\right)^{1/2}$	$k=1$, $\alpha=2.7$ $G=.172$ 0.133	1.17	1.085



$\alpha=\frac{a_2}{a_1}$

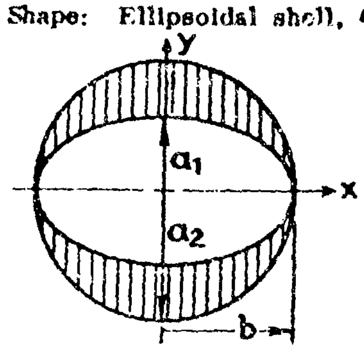
$\beta=\frac{b}{a_1}$

$k=\frac{a_2}{b}$



CASE X (FIGURE 6)

Coil Shape: Ellipsoidal shell, α = constant



$$\alpha = a_2/a_1$$

$$\beta = \frac{h_{max}}{a_1}$$

Current Distribution: $i = i_0/y$

Geometry Factor:

i) $\beta > 1$, $\beta > \alpha$

$$G = \frac{\pi^{1/2}}{5 \ln \alpha} \beta^{1/2} \left(\frac{1}{(\beta^2-1)^{1/2}} \ln(\beta + (\beta^2-1)^{1/2}) - \frac{1}{(\beta^2-\alpha^2)^{1/2}} \ln\left(\frac{\beta + (\beta^2-\alpha^2)^{1/2}}{\alpha} \right) \right)$$

ii) $\beta < 1$, $\alpha > \beta$

$$G = \frac{\pi^{1/2}}{5} \beta^{1/2} \left(\frac{1}{(\beta^2-1)^{1/2}} \sec^{-1} \frac{1}{\beta} - \frac{1}{(\alpha^2-\beta^2)^{1/2}} \sec^{-1} \frac{\alpha}{\beta} \right)$$

iii) $\beta > 1$, $\alpha > \beta$

$$G = \frac{\pi^{1/2}}{5} \beta^{1/2} \left(\frac{1}{(\beta^2-1)^{1/2}} \ln(\beta + (\beta^2-1)^{1/2}) - \frac{1}{(\alpha^2-\beta^2)^{1/2}} \sec^{-1} \frac{\alpha}{\beta} \right)$$

$G_{max} = .204$ at $\alpha = 6$, $\beta = 2$

Current Density Factor J:

$$J = \left(\frac{1}{4 \pi \beta \ln \alpha} \right)^{1/2}$$

at $\alpha = 6$, $\beta = 2$, $G_{max} = 0.204$

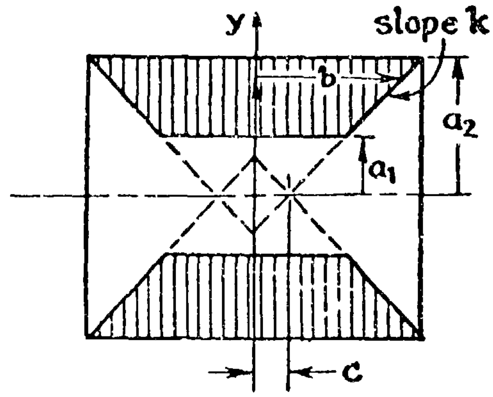
and $J_{max} = 0.149$

Current Density Ratio γ for same Field referred to case VIII: = 1.31

Power Ratio P for same Field referred to case VIII = 0.769

CASE XI

Coil Shape: Cylindrical Bore, tapered ends, non-coinciding apexes.



$b = ky + c$

k = slope

$\gamma = \frac{c}{a_1} \quad \alpha = a_2/a_1$

Current Distribution: $i = i_0/y$

Geometry Factor:

$$G = \frac{\pi^{1/2}}{5} \frac{1}{k(\alpha-1)+\gamma \ln \alpha} \left[\frac{k}{k^2+1} \cdot \right.$$

$$\cdot \ln \frac{((k^2+1)\alpha^2+2k\gamma\alpha+\gamma^2)^{1/2}+\alpha(k^2+1)+\gamma\left(k^2/k^2+1\right)^{1/2}}{((k^2+1)+2k\gamma+\gamma^2)^{1/2}+(k^2+1)^{1/2}+\gamma\left(k^2/k^2+1\right)^{1/2}}$$

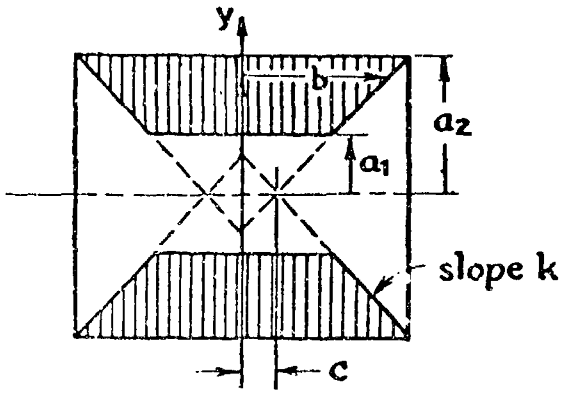
$$\left. + \ln\left(\alpha \frac{((k^2+1)+2k\gamma+\gamma^2)^{1/2} + \gamma + k}{((k^2+1)\alpha^2+2k\gamma\alpha+\gamma^2)^{1/2}+\gamma+k\alpha}\right)\right]$$

G is a maximum when k = 1 and α = 4.5 which reduces case XI to case VII Current Density Factor J:

$$J = \left(\frac{1}{4\pi(k(\alpha-1)+\gamma \ln \alpha)}\right)^{1/2}$$

CASE XII

Coil Shape: Cylindrical Bore, tapered ends, non-coinciding apexes.



$$b = ky + c$$
$$k = \text{slope}$$
$$\gamma = \frac{c}{a_1} \quad \alpha = a_2/a_1$$

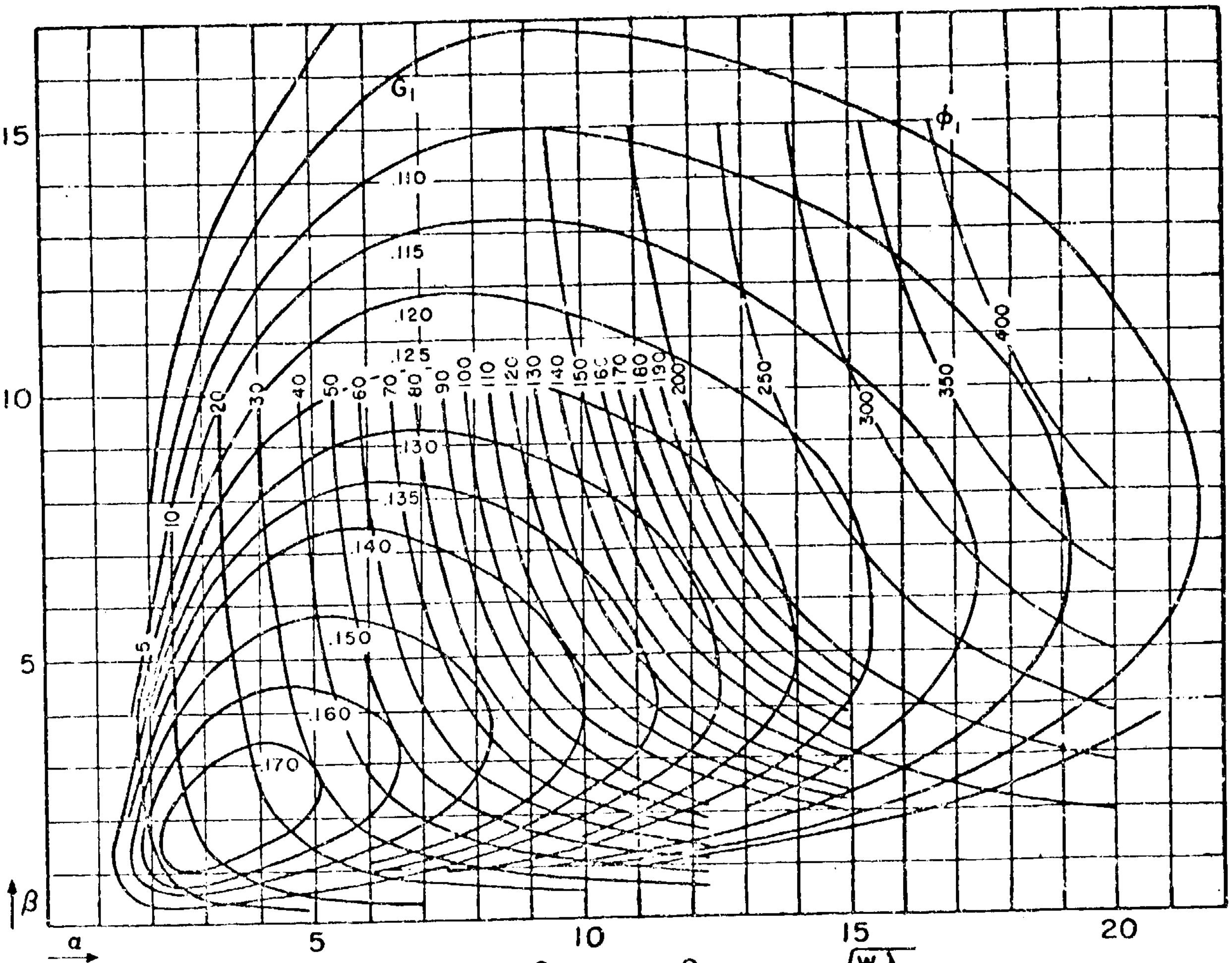
Current Distribution: $$i = i_0/b = \frac{i_0}{ky+c}$$

Geometry Factor:

$$G = \frac{\pi^{1/2}}{5}\frac{k}{(k^2+1)^{1/2}}\left[\left(\frac{1}{k(\alpha-1)+\gamma \ln\left(\frac{\gamma+k}{\gamma+\alpha k}\right)}\right)\cdot\right.$$
$$\left.\cdot \ln \frac{((k^2+1)\alpha^2+2k\gamma\alpha+\gamma^2)+\alpha(k^2+1)^{1/2}+\gamma\left(\frac{k^2}{k^2+1}\right)^{1/2}}{((k^2+1)+2k\gamma+\gamma^2)^{1/2}+(k^2+1)^{1/2}+\gamma\left(\frac{k^2}{k^2+1}\right)^{1/2}}\right]$$

G is a maximum when $k = 1.0$ and $\alpha = 4.5$ which reduces case XII to case VII Current Density Factor J:

$$J = \left(\frac{1}{4\pi\left(k(\alpha-1)+\gamma \ln\left(\frac{k+\gamma}{k\alpha+\gamma}\right)\right)}\right)^{1/2}$$



$$L/R = a_1^2\,\lambda/\rho.\,10^{-9}\phi_1 \quad .\,B = G_1\sqrt{\frac{w\lambda}{\rho a_1}}$$

Fig. I Reproduction of Cockcroft's G-factor curves for uniformly wound coils with a square cross-section (Case VIII)

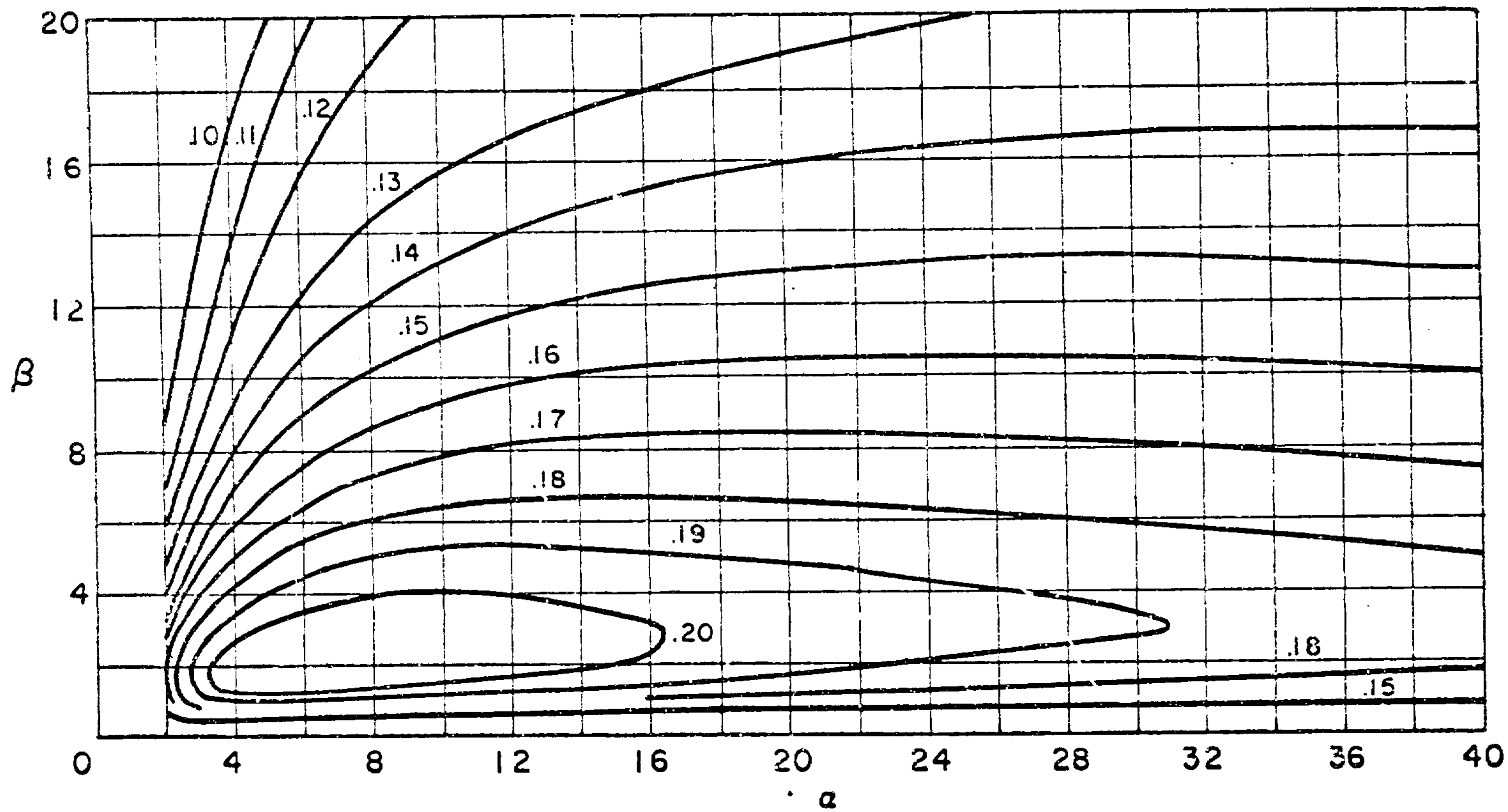


Fig. 2 Reproduction of Bitter's G-factor curves for plate magnets, $i = i_0/r$ (Case VI)

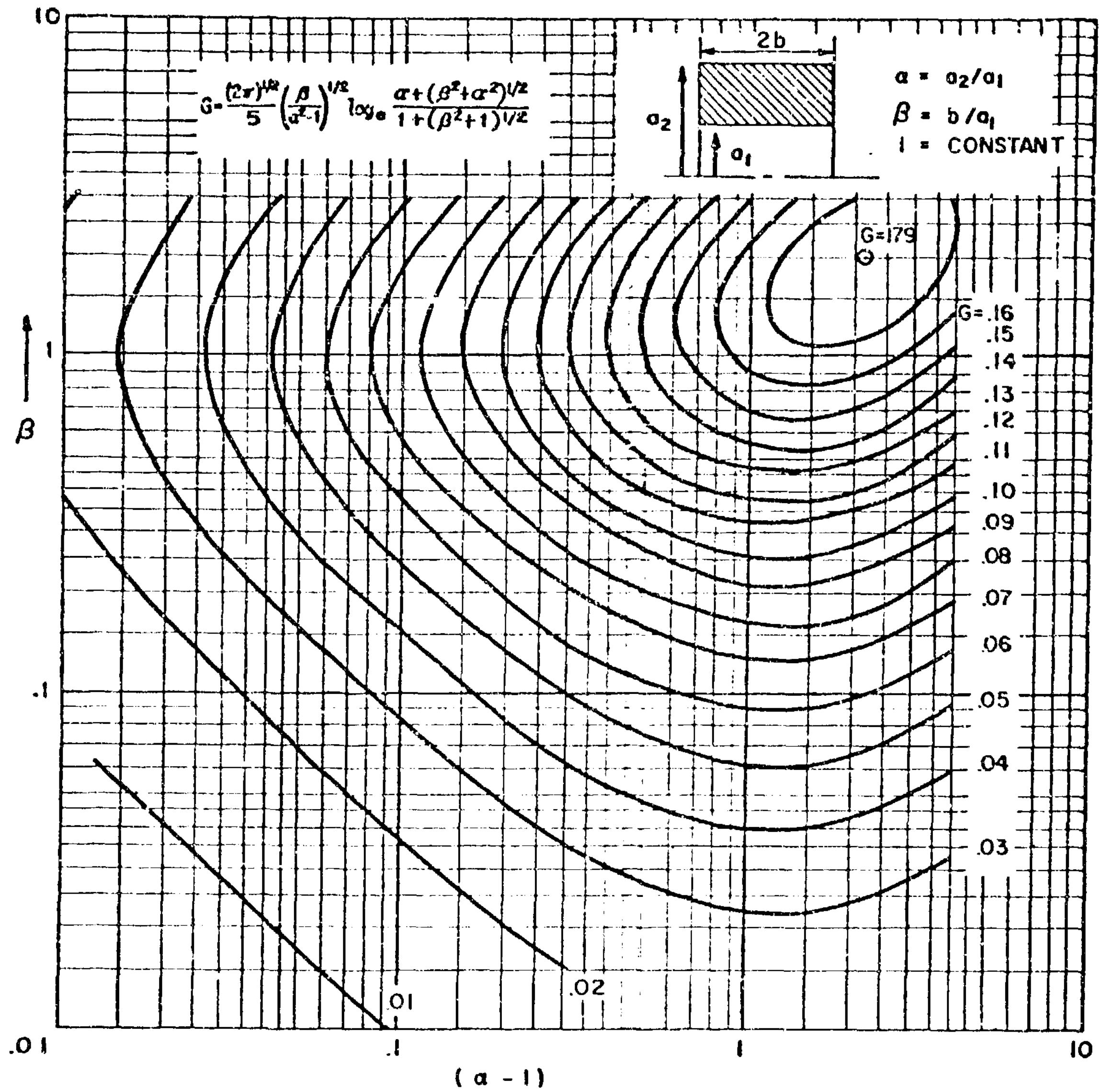


Fig. 3 G-factors for uniform current density magnets with small α and β's (Case VIII)

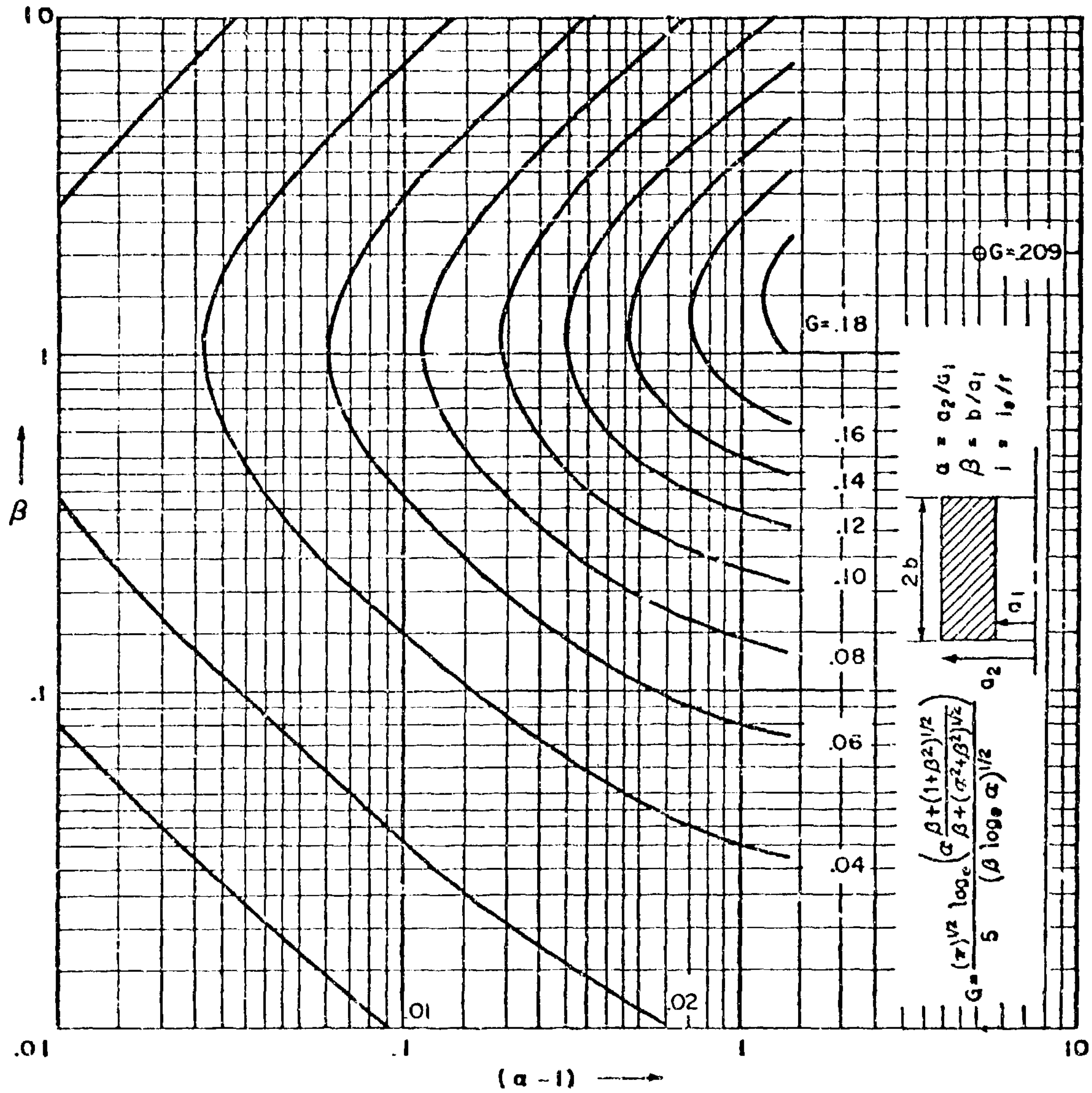


Fig. 4 G-factors for radial current density magnets with small α and β's (Case VI)

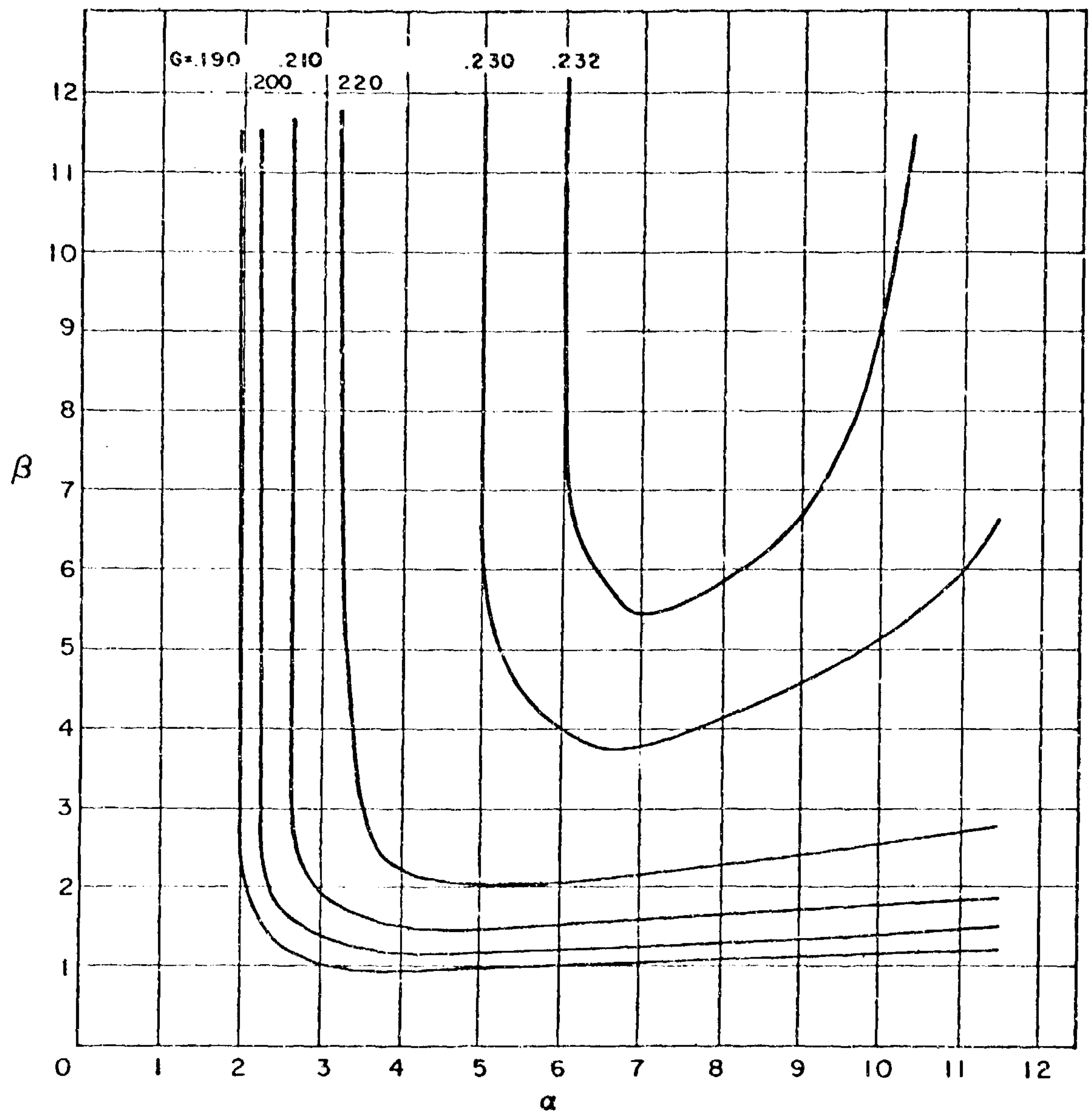


Fig. 5 G-factors for a Gaume current distribution (Case V)

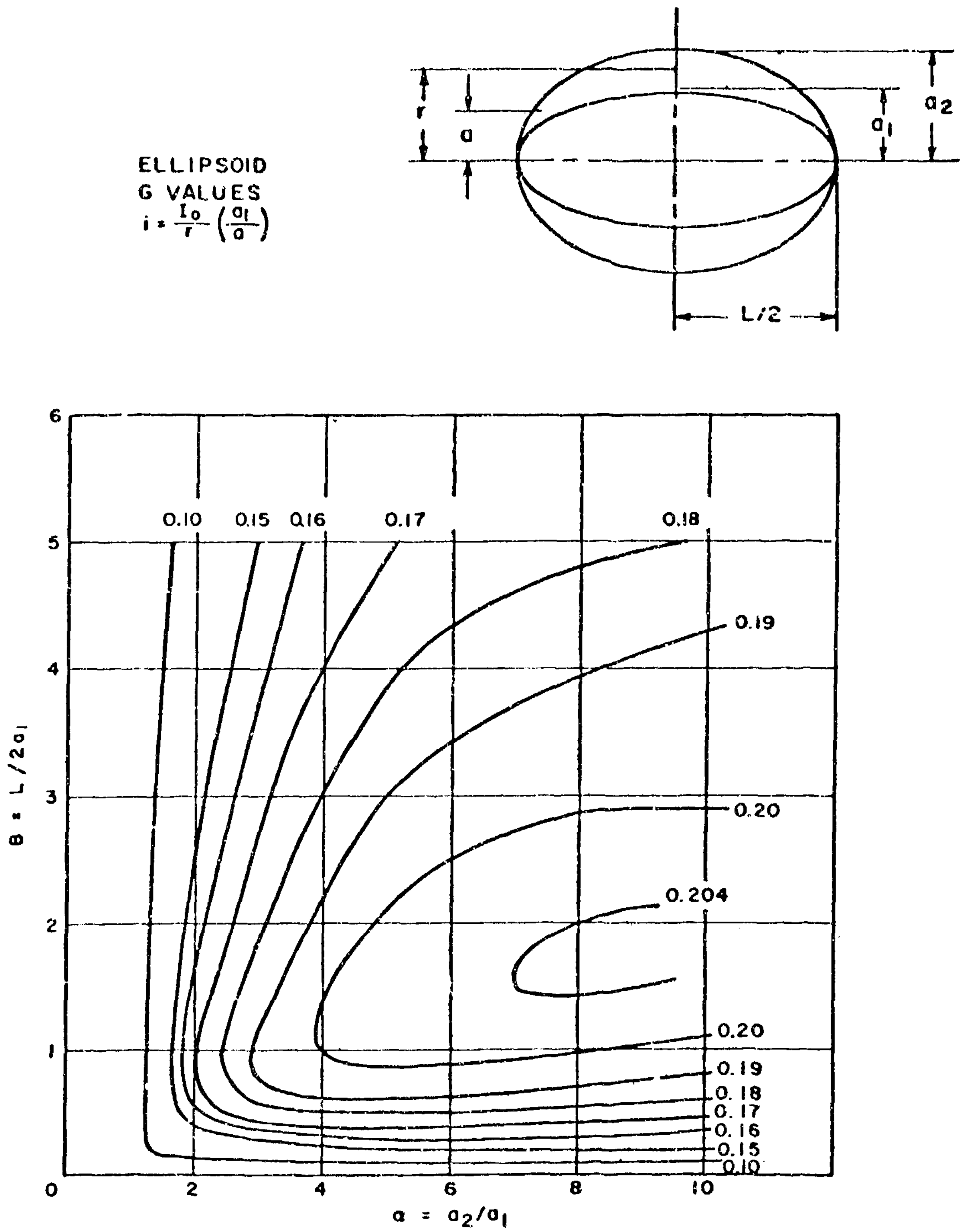


Fig. 6 G-factors for an ellipsoid of constant α (Case X)

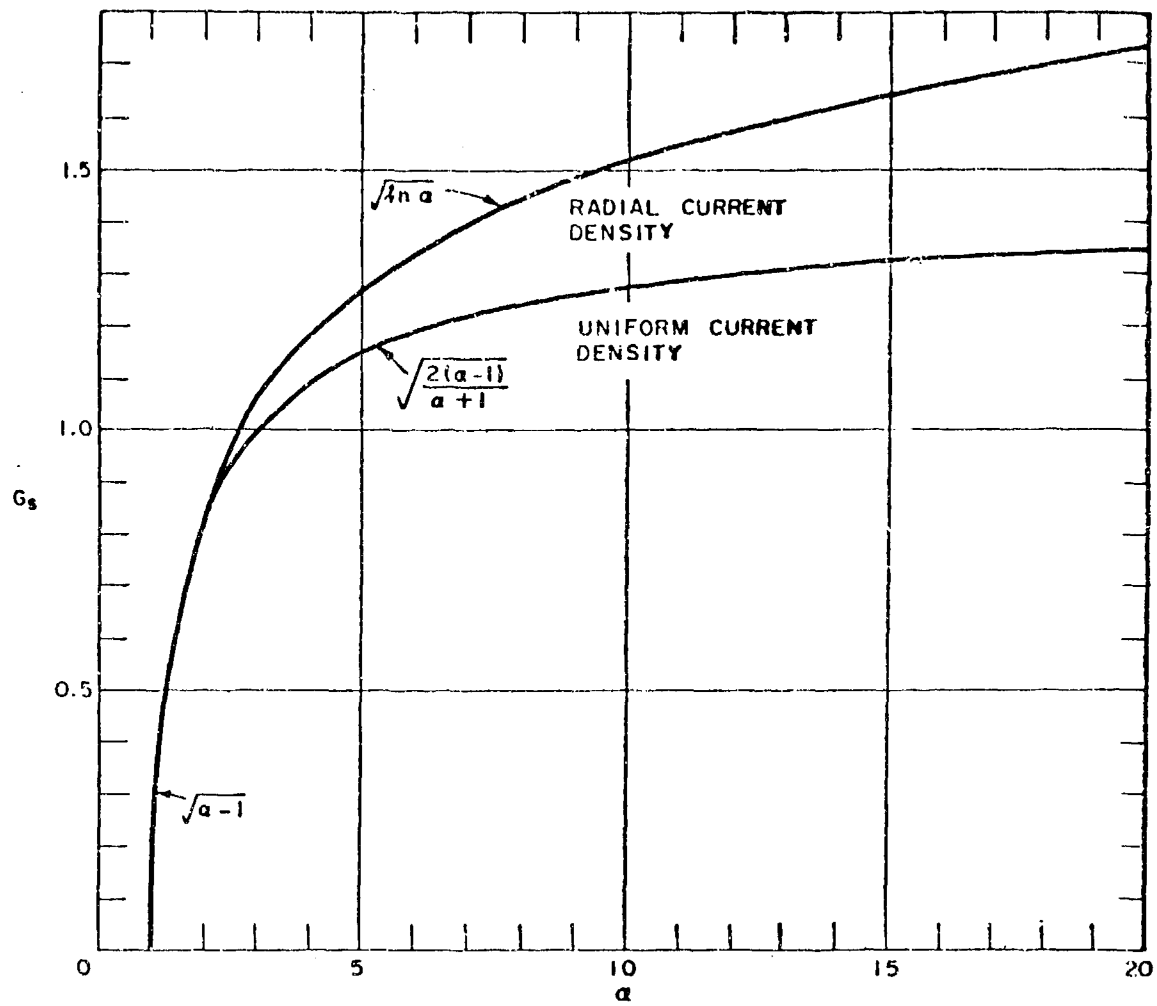


Fig. 7 G-factors for long solenoids

PART II

HOMOGENEOUS MAGNETIC FIELDS

A.

A magnetic field can be written as a power series involving Legendre polynomials expanded about the origin of the field. If this origin is on the axis and on the plane of symmetry the expansion will have no odd terms and the axial component H_z and the radial component H_ρ can be written:

$$H_z(r,\theta) = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!} \left[H_z^{(2n-2)}(z,0) \right] r^{2n-2} P_{2n-2}(u) \tag{1}$$

$$H_\rho(r,\theta) = -\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} \left[H_z^{(2n-2)}(z,0) \right] r^{2n-2} P'_{2n-2}(u) \tag{2}$$

$$u = \cos\theta$$

In these equations the coefficients are defined as follows:

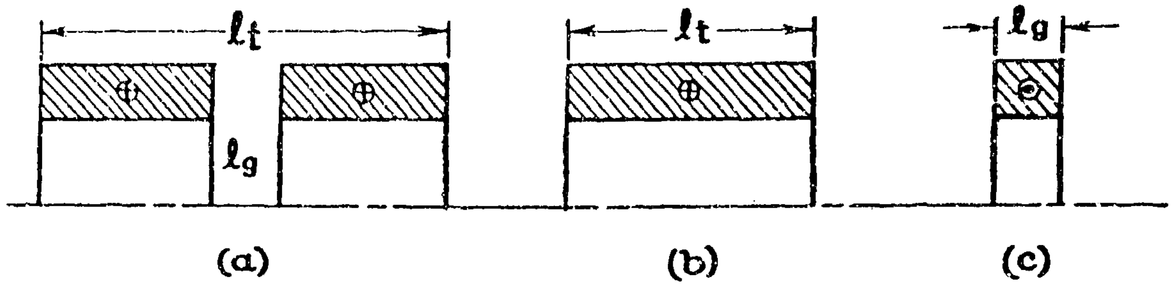
$$H_z^{(2n-2)}(z,0) \equiv \left. \frac{d^{2n-2} H_z(z,0)}{dz^{2n-2}} \right]_{z=0}$$

and $P_{2n-2}(u)$ are the Legendre polynomials tabulated in Table 2.

The importance of this approach, due to W. Garrett[1], is that the field at any point on the axis of a system of coaxial coils can be given explicitly in simple algebraic form. From this, the derivatives of equations (1) and (2) can be obtained, and so a complete expression for the field in the vicinity of the origin can be obtained. If sufficient derivative terms are used the field can be found anywhere within a sphere whose center lies at the origin and which does not extend far enough to include any corners of the coil. However, out to within a few percent of the inner radius of the coil, the field can be found quite accurately with only a few derivative terms. Compensation can be designed for each component of the field separately. In general, if we have a coil system of the assumed symmetry having N variable parameters such as size, or ampere-turns, it is possible to specify N coefficients in the expansions 1 and 2.

For example, to examine the standard Helmholtz pair of coils, where the coils are so spaced that there is no second derivative of the field at the origin, we can think of the coil system in the following way:

1. M. W. Garrett, J. Appl. Phys. 22, 9, Sept. 1951



$a = b - c$

Figure 1

As shown in Fig. 1, we consider a single coil having length equal to the coils and the gap separation, from which we subtract a second coil of the length of the gap. The gap coil is thought of as having a current in the reverse direction to the original long coil. We now merely need select the length of the gap coil so that it will have a second derivative equal and opposite to that of the original long coil and the two will thus cancel at the origin.

There are many other combinations which would also cancel the second derivative at the origin, for instance, the hypothetical gap coil with its reverse current could have been somewhat longer and had less reverse current in it, or shorter and had more reverse current in it. Both of these solutions, of course, lead to a continuous solenoid with no gap but with an altered current density in the center section. These various solutions will have both an effect on the amount of field in the center of the magnet and on the size of the higher derivatives. The greater the distance over which one compensates, the more parallel will be the flux lines from the compensating coil, hence the flatter will be the flux distribution at the origin, and hence the smaller will be the sum of the higher derivatives. This compensation may not be compatible with obtaining the highest central field for the least power, however.[2]

B.

Equations (1) and (2) can be rewritten in the following form:

$$H_z(r,0) = H_z(0,0)\left[1+\mathcal{E}_2\left(\frac{r}{a_1}\right)^2 P_2(u) + \mathcal{E}_4\left(\frac{r}{a_1}\right)^4 P_4(u) + \cdots\right] \quad (3)$$

$$H_\rho(r,0) = H_z(0,0)\left[0+\mathcal{E}_2\left(\frac{r}{a_1}\right)^2 P_2'(u) + \mathcal{E}_4\left(\frac{r}{a_1}\right)^4 P_4'(u) + \cdots\right] \quad (4)$$

$$H_z(0,0) = G\left(\frac{W\lambda}{\rho a_1}\right)^{1/2}$$

2. R. S. Ingarden and J. Michalczyk, Bulletin De L'Academie Polonaise Des Sciences (Serie des sci. math, astr., and phys) VIII, 5, 1960.

The expansion has been normalized to the inside radius of the coils a_1. A tabulation of the $\mathcal{E}_n$ coefficients for three useful geometries is given in Table 1. The geometries treated are (i) a cylindrical thin current sheet, (ii) a finite solenoid with a uniform current density, and (iii) a finite solenoid with a radial current distribution ($i = {}^{i_0}/_r$). The value of the first eight Legendre polynomials and their derivatives are given for convenience in Table 2. From equations (3) and (4) the magnetic field can be found at any point near the origin of a magnet, at a distance r from the origin and at an angle θ. The derivative terms in equation (3) have their largest value for $\theta = 0$ ($P_n(1) = 1$ for all n when $\theta = 0$ and $\cos\theta = 1$). The maximum value of the terms therefore can be obtained from the simpler form of (3) where $\theta = 0$ and $z = r$:

$$H_z(Z,0) = H_z(0,0)\left[1 + \mathcal{E}_2\left(\frac{z}{a_1}\right)^2 + \mathcal{E}_4\left(\frac{z}{a_1}\right)^4 + \cdots\right] \tag{5}$$

C.

The most commonly used method of achieving some degree of homogeneity in coils is to use the spaced helmholtz pair. Garrett has compiled a number of tables dealing with this method of compensation and since these tables have not been published we are making them part of this report. These tables are for constant current density only. The first table, Table 3 is used to determine the proper spacing between finite coils to achieve cancellation of the second derivative at the origin. The second table, Table 4 is used to determine how far from the origin one can move before the field deviates by more than 0.1 percent of the field at the origin. The third table, Table 5 is particularly useful and can show how accurately the coils must be spaced in order to cancel the second derivative to the desired degree. The two remaining tables, Table 6 and Table 7 give the resultant fourth and the sixth $\mathcal{E}$ coefficients remaining after the proper spacing of the coils. The tables are used with the notation of equations (3) and (4) except that Garrett has normalized the expansion to the mean radius of the coil rather than to the inner radius as in Section B above ($(\frac{z}{a_0})$ rather than $(\frac{z}{a_1})$ where $a_0 = (\frac{a_1 + a_2}{2})$).

To illustrate the use of Garrett's tables, we work out the following examples:

Assume we have two coils, each with an $\alpha = 3$ and a $\beta = 1$ and we wish to separate them to cancel the second order derivative at the origin.

$$X = \frac{4\beta}{\alpha+1} = 1.0$$

$$A = \frac{2(\alpha-1)}{\alpha+1} = 1.0$$

From Table 3, $K = .58250 = \dfrac{\Delta X}{a_1 + a_2} = \dfrac{\Delta X}{a_1(1+\alpha)}$

therefore $$\frac{\Delta X}{a_1} = 2.33040 = 2\beta + \frac{l_g}{a_1}$$

and the spacing should be

$$\frac{l_g}{a_1} = .33040$$

Using Table 4 we see that $V = \frac{z}{a_0} = .181$ or $\frac{z}{a_1(1+\alpha)} = .181$ and $\frac{z}{a_1} = .724$. This means that out to a distance along the axis of 72.4% of the inner radius, the field is within 0.1% of the field at the origin. Using Table 5, we can examine the effect of improper spacing.

$$H = H_0\left(1 + Q\left(\frac{d}{a_0}\right)\cdot\left(\frac{z}{a_0}\right)^2 + \mathcal{E}_4\left(\frac{z}{a_0}\right)^4 + \cdots\right)$$

where Q is the value in table 5 and d/a_o is the displacement error in locating one coil. For the above example, we can find the error that will make the uncancelled 2nd derivative error equal the fourth at $\left(\frac{z}{a_0}\right) = 0.1$

$$Q = 4.21 \ (\text{at } X = 1, A = 1)$$
$$\mathcal{E}_4 = .922 \text{ from table 6}$$
$$4.21\left(\frac{d}{a_0}\right) \times 10^{-2} = .922 \times 10^{-4}$$
$$\frac{d}{a_0} = 2.18 \times 10^{-3}$$

D.

When maximum homogeneity and not maximum efficiency is of the greatest importance, the higher derivatives must be handled. Since the number of derivatives that can be cancelled depends upon the number of variables, we must now pick solutions which have more variables at our disposal. For instance, if we wish to cancel both the second and the fourth derivative of the field we can superimpose two coils on each other each with two geometry variables or each with one geometry variable and one current variable. To proceed, then, we would pick the length of one coil and the current of one coil and the variables then would be the length of the second coil and the ratio of the two current densities. Garrett has calculated a set of so called sixth-order solenoids (no second or fourth derivatives) by making the following choice of variables: He constructs an overwound end solenoid by winding N layers of turns over a certain central region of the solenoid and 2N layers over a certain section of both ends (i.e. a notch on the center section of the o.d. of the coil). He now solves uniquely for the length of the overall coil and the length of the notch. The results of his calculation are shown in Table 8 for several thicknesses of coil.

There are other choices of two variables that one can make to construct sixth-order solenoids and Table 9 illustrates several methods. Table 9 was constructed for the specific example of a rather long solenoid. Several general remarks can be drawn from the table. Probably the most important of which is that since the solenoid is very long

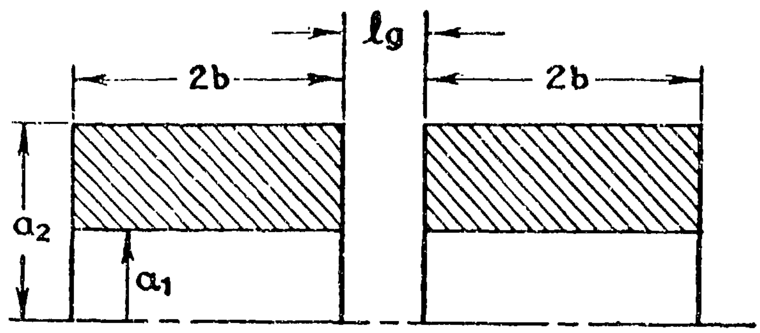
the job of compensating it for high homogeneities is much simplified, the higher derivative having been quite small to begin with. We notice, however, that in all but one case in the table, cancelling the second and fourth derivatives of the field, have increased the sixth derivative. The single exception is the current sheet of the o.d. The current sheet on the o.d., however, requires more power than any of the other methods of solution. Perhaps the most attractive method of solution is that represented by the last column, that of current sheets on either end of the i.d. of the coil. This method takes little power and leaves the center section of the magnet open for further compensation if desired.

Cancellation of higher derivatives than the fourth is of course possible, but is often of decreasing practical importance because of the unrealistic accuracy necessary in the cancellation of lower derivatives, if the leftover uncompensated part of say the second, is to be smaller than the sixth derivative or eighth derivative, etc. It probably makes sense to design coils to cancel the second and the fourth derivative and use a small auxiliary set of coils driven from the separate power source to tune out any random errors which creep in. Some of these random errors are of course going to be nonsymmetrical and will give rise to odd derivative terms. It is quite possible to construct a set of auxiliary compensating coils which will operate independently of each other. One coil can be designed to have a second derivative and no fourth and a second coil to have a fourth derivative and no second, etc. Each one can then be tuned independently of the other.

E.

We have constructed a table of $\mathcal{E}$ coefficients up to the eighth order for coils of varying α and β with both constant current density and with current density equal $i = i_0/p$. The tables also include the G factor and the multiplication of the G factor and the $\mathcal{E}$ coefficients. The tables are to be used with the expansion of equation (3) and (4) where z is referred to the inner radius a_1; $\mathcal{E}$ is tabulated as D for uniform currents and as A for the radial current distribution. Table 10A gives the coefficients for the uniform current density case for several α's and for β up to 10. Table 10B gives the coefficients over the same range of α and β for the radial current distribution case. Tables 11A, B, C, D show more detail for the constant current and the radial current cases in the small β range.

The use of the tables is probably best illustrated by the following example: Let us assume we have a coil we wish to separate for purposes of access and we want to know the resultant field in the gap and the resultant homogeneity (see Fig. 2)



W = power
λ = space factor
ρ = resistivity

$$\alpha = a_2/a_1$$

$$\beta_t = \frac{4b+\ell_g}{2a_1} \qquad G_t(\alpha, \beta_t)$$

$$\beta_g = \frac{\ell_g}{2a_1} \qquad G_g(\alpha, \beta_g)$$

Figure 2

If we now subtract from the field expansions for a coil of length $4b + \ell_g$, a coil of length ℓ_g we can write:

$$H_{gap} = (H_{0t} - H_{0g}) + (H_{2t} - H_{2g})\left(\frac{z}{a_1}\right)^2 + (H_{4t} - H_{4g})\left(\frac{z}{a_1}\right)^4 + \cdots$$

$$H_{gap} = \left(\frac{W\lambda}{\rho a_1}\right)^{1/2}\left(\frac{\beta_t}{\beta_t - \beta_g}\right)^{1/2}\left[\left(G_t - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}}\right) + \left(G_t \mathcal{E}_2 - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}} \mathcal{E}_2'\right)\left(\frac{z}{a_1}\right)^2 + \right.$$

$$\left.\left(G_t \mathcal{E}_4 - \frac{G_g \beta_g^{1/2}}{\beta_t^{1/2}} \mathcal{E}_4'\right)\left(\frac{z}{a_1}\right)^4 + \cdots\right.$$

where the primed quantities refer to the error coefficients for the gap. Taking a specific geometry, let the current be constant ($\mathcal{E}$ = D).

let

$$\beta_t = 4$$
$$\alpha = 3$$
$$\beta_g = 0.2$$

What is the field in the gap and the second error coefficient, and what would be the second error coefficient if there were no access gap?

The D_2 listed for $\alpha = 3$, $\beta = 4$ in Table 10A is $-.1415 \times 10^{-1}$, therefore the second error coefficient of the whole coil would be $\mathcal{E}_2 = -1.415 \times 10^{-2}$ where

$$H = H_0 \left(1 + \mathcal{E}_2 \left(\frac{z}{a_1}\right)^2 + \cdots\right)$$

To solve for the split coil, we use (6) and the G_1. D_n columns in Table 10A

$$H_{gap} = \left(\frac{W\lambda}{\rho_1 a_1}\right)^{1/2} \left(\frac{4}{3.8}\right)^{1/2} \left[\left(0.1580 - 0.08639\left(\frac{0.2}{4}\right)^{1/2}\right) + \right.$$

$$\left. + \left(-0.2235 \times 10^{-2} + 0.5004 \times 10^{-1}\left(\frac{0.2}{4}\right)^{1/2}\right)\left(\frac{z}{a_1}\right)^2 + \cdots\right]$$

$$H_{gap} = \left(\frac{W\lambda}{\rho a_1}\right)^{1/2} (1.03) \left[.1387 + 0.8965 \times 10^{-2} \left(\frac{z}{a_1}\right)^2\right]$$

this gives a central field of

$$H_{gap}(0,0) = \left(\frac{W\lambda}{\rho a_1}\right)^{1/2} (0.143)$$

and if the second order coefficient is written in the following form:

$$H = H_0 \left(1 + \mathcal{E}_2 \left(\frac{z}{a_1}\right)^2 + \cdots\right)$$

the coefficient for the above example would be

$$\mathcal{E}_2 = \frac{0.8965 \times 10^{-2}}{.1387} = 6.46 \times 10^{-2}$$

The loss in field for this case has been 9.5% and the second order error coefficient has been increased by a factor of 4.55. If we had been trying to homogenize the field by separating the coils, we note that we would have separated them by too much. To use these tables for synthesis (i.e. cancellation of derivatives) it would be necessary to use trial and error, in the manner of the above example.

It is interesting to note the oscillatory behavior of the $\mathcal{E}$ coefficients, particularly in the small β region. The second derivative is always negative, the fourth derivative goes through zero once, the sixth derivative goes through zero twice and the eighth derivative three times. This means that there are coils which can be built with no fourth, no sixth and no eighth derivatives. This may be useful in the design of independent compensating coils.

F.

If one wishes to compensate a very long solenoid over a considerable length of the axis such as a multicoil structure used for plasma experiments, expansion of the derivatives around the origin has limited usefulness. One might wish to use the following procedure instead; namely specifying that the field at any number of intervals along the axis be constant and then solving for the current distribution in the coils that would make the field be equal at the specified points. In the case of a multicoil structure one could ask that the total field in the center of each of the coils be equal to a constant and then solve for the current density required in each of the coils to make this true. The same procedure could be used with a single long solenoid where the solenoid could be broken up into an arbitrary number of pieces.

The most useful method of solution of this type of problem is basically the following: taking any single coil we first find the field at the center of that coil and at distances away from the coil represented by the center lines of each of the other coils, for a given current. Then by properly adding up the coils in matrix form and demanding that the field matrix times the current matrix be a constant we can invert the current matrix and solve for the individual currents required in each of the coils to achieve the desired results. The method of solution and a useful notation are developed in the following example. Mr. R. Bradshaw of Conesco, Inc., Arlington, brought this method of solution to our attention. Consider the following system of coils arranged so that they have a common axis. They may in general be of various sizes but for illustration consider them to be all identical with the same separation. For ease in calculation choose an even number of coils so that we have a plane of symmetry passing between coils.

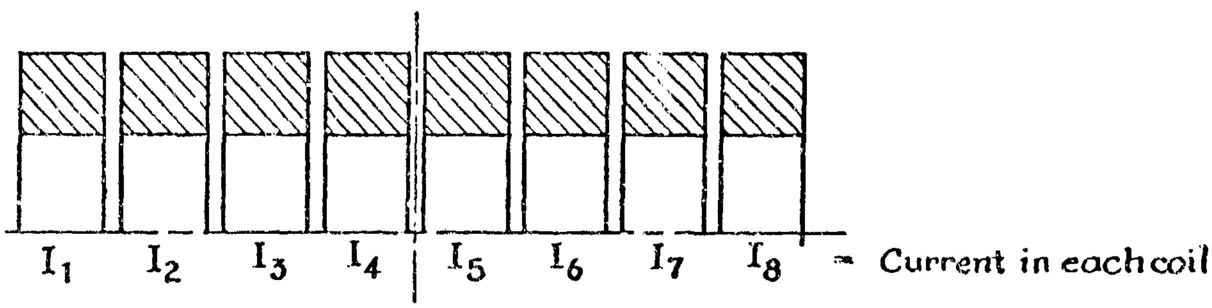


Figure 3

Our object is to find what ratio of currents is needed in the various coils so that a uniform magnetic field is obtained throughout the interior. If the coils are thin and close together, to a good approximation uniformity is obtained by requiring that the field at the center of each coil be the same. The field falls off rapidly on each side of the center of a single coil and we may construct a matrix the elements of which represent the contributions of the coils at their respective centers. Let F_{ij} be the field at the center of coil i due to the field of coil j. For example, for 4 coils, 2 coils on each

side of the plane of symmetry, we have

$$\left|\begin{array}{cc|cc} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ \hline F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{array}\right|$$

These matrix elements may be expressed in % of an infinite solenoidal sheet of the same mean radius, or whatever one desires, since we need only ratios.

$F_{ij}\, I_j = \mathcal{F}_i$ where $\mathcal{F}_i$ is the field at the center of the i^{th} coil or

$$\left|\begin{array}{cc|cc} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ \hline F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{array}\right| \left|\begin{array}{c} I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}\right| = \left|\begin{array}{c} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \end{array}\right|$$

If one now specifies the $\mathcal{F}_i$'s the F_{ij} matrix can be inverted and the I_j matrix found.

If the current in the coils are arranged so that $I_1 = I_4$ and $I_2 = I_3$ then it is convenient to partition the matrix as follows:

$$f_{11} = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix},\quad f_{12} = \begin{vmatrix} F_{13} & F_{14} \\ F_{23} & F_{24} \end{vmatrix},\quad f_{21} = \begin{vmatrix} F_{31} & F_{32} \\ F_{41} & F_{42} \end{vmatrix},\quad f_{22} = \begin{vmatrix} F_{33} & F_{34} \\ F_{43} & F_{44} \end{vmatrix}$$

$$i_1 = \begin{vmatrix} I_1 \\ I_2 \end{vmatrix},\quad i_2 = \begin{vmatrix} I_3 \\ I_4 \end{vmatrix};\quad \mathcal{F}_1' = \begin{vmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{vmatrix}\quad \mathcal{F}_2' = \begin{vmatrix} \mathcal{F}_3 \\ \mathcal{F}_4 \end{vmatrix}$$

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \begin{vmatrix} i_1 \\ i_2 \end{vmatrix} = \begin{vmatrix} \mathcal{F}_1' \\ \mathcal{F}_2' \end{vmatrix}$$ and then where $i_2 = k i_1$ and $\mathcal{F}_2' = k\mathcal{F}_1'$, $k = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = i_2 \quad \text{and} \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \begin{vmatrix} i_1 \\ k i_1 \end{vmatrix} = \begin{vmatrix} \mathcal{F}_1' \\ k\mathcal{F}_1' \end{vmatrix}$$

it follows that $(f_{11} + f_{12} k)\, i_1 = \mathcal{F}_1'$

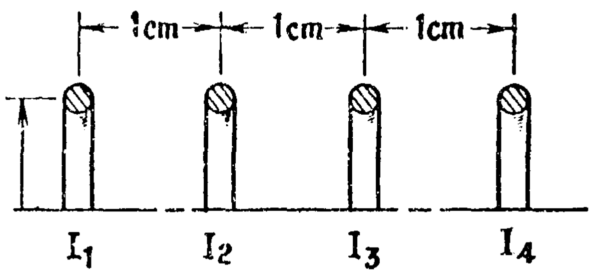
Since we desire the fields at the centers of each coil to be a constant

$$\mathcal{F}_1 = \mathcal{F}_2 = \text{constant} = C \qquad \text{where } C = c\begin{vmatrix}1\\1\end{vmatrix}$$

$$\text{i.e., } (f_{11} + f_{12}k)\,i_1 = C$$

It is clear that $i_1 = (f_{11} + f_{12}k)^{-1}C$ and this gives us the desired ratio of currents.

As an example consider the field on the axis due to a system of four current loops of radius $y = 1\,\text{cm}$



$I_1 = I_4$

$I_2 = I_3$

Figure 4

Separation between centers of the coils is 1 cm. We need calculate the field of one loop at the center and 1, 2, and 3 cm. on the axis away from the center. For a single loop

$$H_x = \frac{2\pi i}{10}\,\frac{y^2}{(x^2+y^2)^{3/2}}$$

x	H_x/i
0	.628
1	.222
2	.0562
3	.0199

then $$f_{11} = \begin{vmatrix}0.628 & 0.222\\ 0.222 & 0.628\end{vmatrix} \qquad f_{12} = \begin{vmatrix}0.056 & 0.019\\ 0.222 & 0.056\end{vmatrix}$$

$$f_{12}k = \begin{vmatrix}0.056 & 0.019\\ 0.222 & 0.056\end{vmatrix}\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix} = \begin{vmatrix}0.019 & 0.056\\ 0.056 & 0.222\end{vmatrix}$$

$$f_{11} + f_{12}k = \begin{vmatrix}0.628 & 0.222\\ 0.222 & 0.628\end{vmatrix} + \begin{vmatrix}0.019 & 0.056\\ 0.056 & 0.222\end{vmatrix} = \begin{vmatrix}0.647 & 0.278\\ 0.278 & 0.850\end{vmatrix}$$

We now need the inverse of $(f_{11} + f_{12}k)$

By definition, the inverse of a matrix A with matrix elements a_{ij} is obtained if for each element $(a_{ij})^{-1} = \dfrac{(-1)^{i+j}A_{ji}}{\det A}$; A_{ji} is the minor of the determinant A.

If $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ then $A^{-1} = \begin{vmatrix} (a_{11})^{-1} & (a_{12})^{-1} \\ (a_{21})^{-1} & (a_{22})^{-1} \end{vmatrix}$

For 2 x 2 matrices this is very simple:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\det (f_{11} + f_{12} k) = (.850)(.647) - (278)^2 = .473$$

$$(f_{11} + f_{12} k)^{-1} = \frac{1}{.473} \begin{vmatrix} .850 & -.278 \\ -.278 & .647 \end{vmatrix} = \begin{vmatrix} 1.797 & -.587 \\ -.587 & 1.367 \end{vmatrix}$$

$$\therefore \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} 1.797 & -587 \\ -.587 & 1.367 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} c = c \begin{vmatrix} 1.210 \\ .780 \end{vmatrix}$$

$$\therefore I_1 = c(1.210) \quad I_2 = c(0.780)$$

The current in coil 1 must then be 1.55 times greater than the current in coil 2.

Ingarden and Michalczyk[2] have also considered the problem of homogeneities over a certain length of the axis rather than strictly at the origin. They have used the requirement that the mean square deviation over a certain interval be minimized. Their notation is useful, but solution of the equations, which are transcendental in character, presents a number of problems.

G.

It is often of interest to know the field along the axis of a solenoid both inside the solenoid and outside the solenoid. The following two equations can be used to find the field along the axis of a uniform current solenoid and a radially distributed current solenoid (Bitter discs) respectively.

(i) Uniform current $k = z/a_1$ (7)

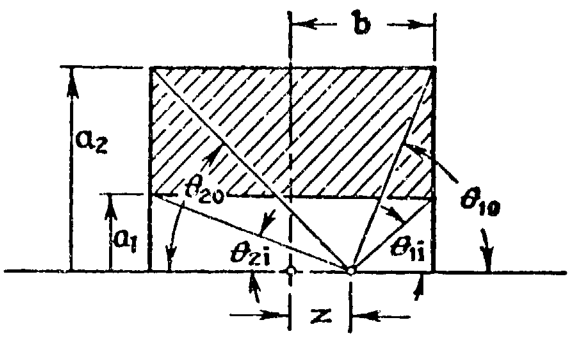
$$H = \frac{\pi i \lambda a_1}{5} \left[\ln \frac{(k-\beta) + (\alpha^2 + (k-\beta)^2)^{1/2}}{(k+\beta) + (\alpha^2 + (k+\beta)^2)^{1/2}} - \ln \frac{(k-\beta) + (1 + (k-\beta)^2)^{1/2}}{(k+\beta) + (1 + (k+\beta)^2)^{1/2}} \right]$$

(ii) Radial current $i = \frac{i_1 a_1}{r}$; $k = \frac{z}{a_1}$

$$H = \frac{\pi i_1 \lambda a_1}{5} \left[(k+\beta) \ln \left(\frac{\alpha + (\alpha^2 + (k+\beta)^2)^{1/2}}{1 + (1 + (k+\beta)^2)^{1/2}} \right) - (k-\beta) \ln \frac{\alpha + (\alpha^2 + (k-\beta)^2)^{1/2}}{1 + (1 + (k-\beta)^2)^{1/2}} \right] \quad (8)$$

Tabulated values of equation (7) for a wide variety of coil shapes have been published by D. E. Mapother and James Snyder in a University of Illinois circular #66 entitled, "The Axial Variation of the Magnetic Field in Solenoids of Finite Thickness". There is no available tabulation of equation (8)

Equations (7) and (8) can be written in a much simpler form utilizing the following angle notation:



$$\frac{a_2}{a_1} = \alpha$$

$$\frac{b}{a_1} = \beta$$

$$\frac{z}{a_1} = \gamma$$

Figure 5

i) uniform current density

$$H = \frac{\pi i_1 \lambda a_1}{5}\left[(\beta-\gamma)\ln\frac{\tan\left(\frac{\pi}{4}+\frac{\theta_{10}}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\theta_{1i}}{2}\right)}+(\beta+\gamma)\ln\frac{\tan\left(\frac{\pi}{4}+\frac{\theta_{20}}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\theta_{2i}}{2}\right)}\right] \tag{9}$$

ii) radial current density

$$H = \frac{\pi i_1 \lambda a_1}{5}\ln\left[\frac{\tan\frac{\theta_{10}}{2}\,\tan\frac{\theta_{20}}{2}}{\tan\frac{\theta_{1i}}{2}\,\tan\frac{\theta_{10}}{2}}\right] \tag{10}$$

General expressions for the off axis fields arising from a loop of current can be written in terms of elliptic integrals of the first and second kind.[3] They can be written in the following way, where the parameters are defined in Figure 6.

$$H_z = \frac{2I}{Q^{1/2}}\left[F(k)+\frac{(a^2-\rho^2-z^2)}{Q}\,s(k)\right] \tag{11}$$

$$H_\rho = \frac{2I}{Q^{1/2}}\left[-F(k)+\frac{(a^2+\rho^2+z^2)}{Q}\,s(k)\right] \tag{12}$$

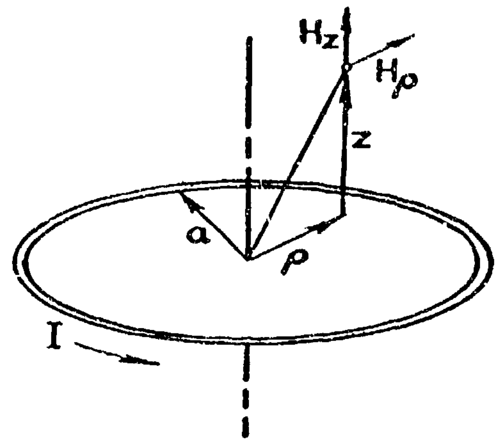
3. W. R. Smythe, STATIC AND DYNAMIC ELECTRICITY, p. 266, McGraw-Hill, New York, 1950

$$Q = (a+\rho)^2 + z^2$$

$$k = \left(\frac{4a\rho}{(a+\rho)^2+z^2}\right)^{1/2}$$

$F(k) =$ Elliptic integral of the first kind

$s(k) = \dfrac{E(k)}{1-k^2}$ where $E(k)$ = elliptic integral of the second kind



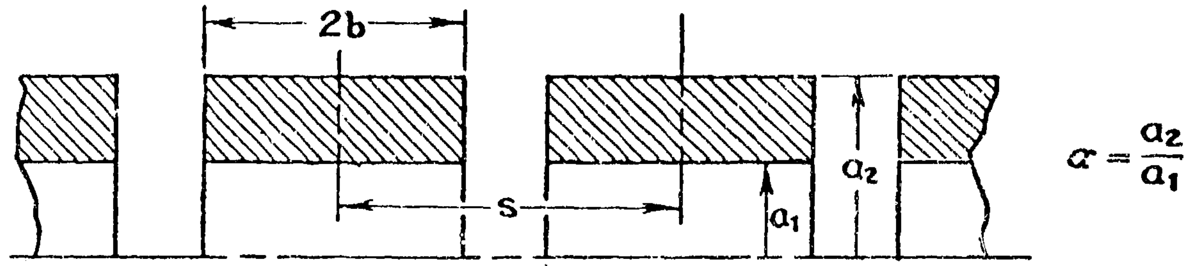
Using equations (11) and (12) the field at any point produced by a finite coil can be found by dividing up the coil into loops and summing all the contributions. A number of laboratories, including our own have written simple computer programs utilizing equations (11) and (12) to find the fields in and outside the conductor volumn.

On-axis and off-axis fields from any solenoid with any radial current distribution can be calculated by means of a lengthy set of tables published by the Oak Ridge National Laboratory Report #ORNL - 2828, entitled, "Tables for a Semi-Infinite Circular Current Sheet," by N. B. Alexander and A. C. Downing. These tables are for the field around the end of a semi-infinite current sheet and the coil in question is built up out of these current sheets in the method shown in the introduction of the table.

Attention is also called to a tabulation by E. E. Callaghan and S. H. Maslen, NASA D465 "The Magnetic Field of a Finite Solenoid". This paper plots graphs of the radial and axial components in and around current sheets of various lengths. While not as accurate as the Oak Ridge tables, it is very useful for approximate results.

A final paper of particular relevance is that of G. R. North, "Some Parameters of Lumped Solenoids", (Oak Ridge ORNL-2975). He derives an expression for the field ripple resulting from the separation of coils in a long multicoil structure (end effects neglected). The equation is as follows (see Figure 7)

$$H(z) = H_{INF}\left(\frac{2b}{s}\right)\left[1 + \left(\frac{s}{a_1}\right)^{1/2}\frac{1}{\alpha-1}\left(e^{-\frac{2\pi a_1}{s}} - \alpha^{1/2}e^{-\frac{2\pi\alpha a_1}{s}}\right)\cdot \cdot\frac{\sin\frac{2b\pi}{s}}{\frac{2b\pi}{s}}\cdot\cos\frac{2\pi z}{s}\right] \qquad (13)$$

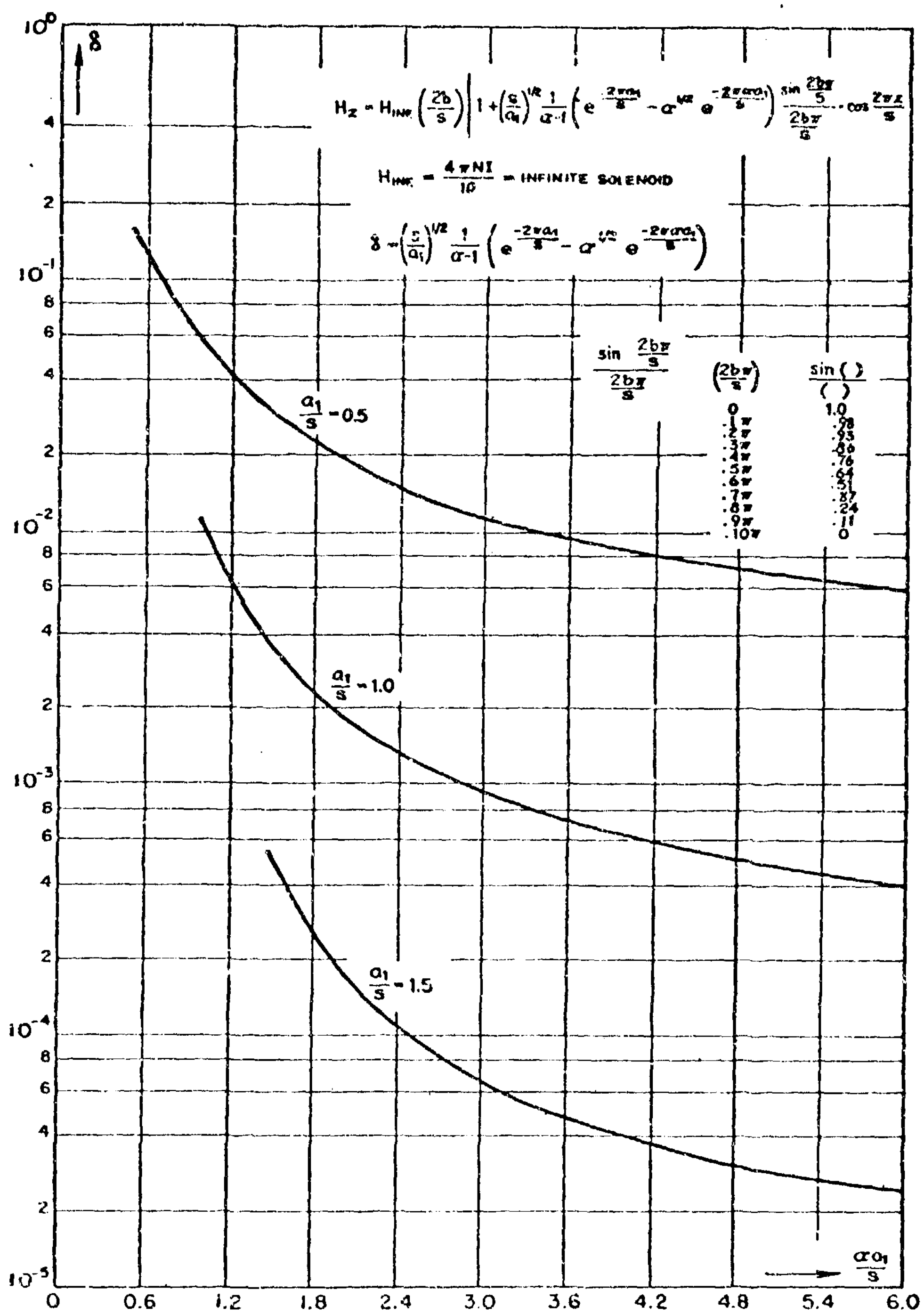


$$H_{INF} = \frac{4\pi NI}{10} = \text{Infinite Solenoid}$$

Figure 7

Figure 8 plots the peak value of the ripple component of eq. (13) for several values of a_1/s. The peak valve occurs when $z = 0$, $\cos\frac{2\pi z}{s} = 1$.

$$\delta = \left(\frac{s}{a_1}\right)^{1/2} \frac{1}{\alpha - 1}\left(e^{-\frac{2\pi a_1}{s}} - \alpha^{1/2} e^{-\frac{2\pi\alpha a_1}{s}}\right) \tag{14}$$



$\left(\frac{2b\pi}{s}\right)$	$\frac{\sin(\ )}{(\ )}$
0	1.0
.1π	.98
.2π	.93
.3π	.86
.4π	.76
.5π	.64
.6π	.51
.7π	.37
.8π	.24
.9π	.11
.10π	0

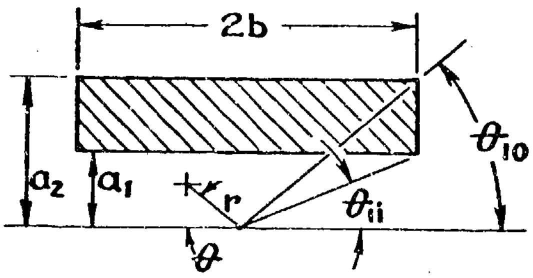
FIG. 8 PEAK VALUE OF FIELD RIPPLE IN MULTICOIL SOLENOIDS
(MULTIPLY BY 10^2 FOR RIPPLE IN PERCENT OF CENTRAL FIELD)

TABLE 1

TABLE OF ERROR COEFFICIENTS

THE $\mathcal{E}$ COEFFICIENTS ARE CONSTRUCTED FROM THE FOLLOWING VARIABLES:

$$c_1 = \frac{1}{1+\beta^2} = \sin^2\theta_{1i}$$



$$c_2 = \frac{\beta^2}{1+\beta^2} = \cos^2\theta_{1i}$$

$$c_3 = \frac{\alpha^2}{\alpha^2+\beta^2} = \sin^2\theta_{10}$$

$$\frac{a_2}{a_1} = \alpha$$

$$c_4 = \frac{\beta^2}{\alpha^2+\beta^2} = \cos^2\theta_{10}$$

$$\frac{b}{a_1} = \beta$$

$$c_5 = \ln\left(\frac{\alpha+(\beta^2+\alpha^2)^{1/2}}{1+(\beta^2+1)^{1/2}}\right) = \ln\left(\frac{\tan(\pi/4+\theta_{10}/2)}{\tan(\pi/4+\theta_{1i}/2)}\right)$$

$$c_6 = \ln\left(\alpha\,\frac{\beta+(\beta^2+1)^{1/2}}{\beta+(\beta^2+\alpha^2)^{1/2}}\right) = \ln\left(\frac{\tan(\theta_{10}/2)}{\tan(\theta_{1i}/2)}\right)$$

THE COEFFICIENTS ARE USED IN THE FOLLOWING SENSE:

$$H_z = H_o\left(1+\mathcal{E}_2\left(\frac{r}{a_1}\right)^2\rho_2(u)+\mathcal{E}_4\left(\frac{r}{a_1}\right)^4\rho_4(u)+\cdots\right)$$

$$u = \cos\theta$$

TABLE 1 (continued)

CASE 1: CURRENT SHEET (I = AMP TURNS/cm.)

$$H(r,\theta) = \frac{4\pi I}{10}\cos\theta\left[1+\mathcal{E}_2\left(\frac{r}{a}\right)^2\rho_2(u)+\mathcal{E}_4\left(\frac{r}{a}\right)^4\rho_4(u)+\cdots\right]$$

$$\mathcal{E}_2 = -\frac{3}{2}\,c_1^2$$

$$\mathcal{E}_4 = -\frac{5}{8}\left(4c_1^2 - 7c_1^4\right)$$

$$\mathcal{E}_6 = -\frac{1}{24}\left(\frac{693}{2}c_2^{\,2} - 315c_2 + \frac{105}{2}\right)$$

CASE 2: UNIFORM CURRENT DENSITY SOLENOID

$$H(0,0) = G\left(\frac{W\lambda}{\rho a_1}\right)^{1/2} = H_0$$

$$H(r,\theta) = H_0\left(1+\mathcal{E}_2\left(\frac{r}{a_1}\right)^2\rho_2(u)+\mathcal{E}_4\left(\frac{r}{a_1}\right)^4\rho_4(u)+\cdots\right)$$

$$\mathcal{E}_2 = \frac{1}{2\beta^2 c_5}\left(c_1^{3/2} - c_3^{3/2}\right)$$

$$\mathcal{E}_4 = \frac{1}{12\beta^4 c_5}\left[c_1^{3/2}\left(1+\frac{3}{2}c_2+\frac{15}{2}c_2^{\,2}\right) - c_3^{3/2}\left(1+\frac{3}{2}c_4+\frac{15}{2}c_4^{\,2}\right)\right]$$

TABLE I

CASE 2 (continued)

$$\mathcal{E}_6 = \frac{1}{30\beta^6 c_5}\left[c_1^{3/2}\left(1+\frac{3}{2}c_2+\frac{15}{8}c_2^2-\frac{35}{4}c_2^3+\frac{315}{8}c_2^4\right)\right.$$

$$\left.-c_3^{3/2}\left(1+\frac{3}{2}c_4+\frac{15}{8}c_4^2-\frac{35}{4}c_4^3+\frac{315}{8}c_4^4\right)\right]$$

$$\mathcal{E}_8 = \frac{1}{56\beta^8 c_5}\left[c_1^{3/2}\left(1+\frac{3}{2}c_2+\frac{15}{8}c_2^2+\frac{35}{16}c_2^3+\frac{315}{16}c_2^4\right.\right.$$

$$\left.-\frac{2079}{16}c_2^5+\frac{3003}{16}c_2^6\right)$$

$$-c_3^{3/2}\left(1+\frac{3}{2}c_4+\frac{15}{8}c_4^2+\frac{35}{16}c_4^3+\frac{315}{16}c_4^4\right.$$

$$\left.\left.-\frac{2079}{16}c_4^5+\frac{3003}{16}c_4^6\right)\right]$$

TABLE 1 (continued)

CASE 3: RADIAL CURRENT DISTRIBUTION $\left(i = {}^{i_o}/_{r} \right)$

$$H(o\,o) = G\left(\frac{W\lambda}{\rho a_1}\right)^{1/2} = H_o$$

$$H(r,\theta) = H_o\left(1 + \mathcal{E}_2\left(\frac{r}{a_1}\right)^2 \rho_2(u) + \mathcal{E}_4\left(\frac{r}{a_1}\right)^4 \rho_4(u) + \cdots\right)$$

$$\mathcal{E}_2 = \frac{1}{2\beta^2 c_6}\left(C_4^{3/2} - C_2^{3/2}\right)$$

$$\mathcal{E}_4 = \frac{1}{4\beta^4 c_6}\left[\frac{35}{14}\left(C_4^{7/2} - C_2^{7/2}\right) - \frac{21}{14}\left(C_4^{5/2} - C_2^{5/2}\right)\right.$$

$$\mathcal{E}_6 = \frac{1}{6\beta^6 c_6}\left[\frac{126}{16}\left(C_4^{11/2} - C_2^{11/2}\right) - \frac{140}{16}\left(C_4^{9/2} - C_2^{9/2}\right)\right.$$

$$\left. + \frac{30}{16}\left(C_4^{7/2} - C_2^{7/2}\right)\right]$$

$$\mathcal{E}_8 = \frac{1}{8\beta^8 c_6}\left[\frac{6235}{240}\left(C_4^{15/2} - C_2^{15/2}\right) - \frac{7161}{240}\left(C_4^{13/2} - C_2^{13/2}\right)\right.$$

$$+ \frac{315}{240}\left(C_4^{11/2} - C_2^{11/2}\right) - \frac{525}{240}\left(C_4^{9/2} - C_2^{9/2}\right)$$

$$\left. - \frac{70}{240}\left(C_4^{7/2} - C_2^{7/2}\right)\right]$$

TABLE 2

TABLE OF LEGENDRE POLYNOMIALS
AND THEIR DERIVATIVES (EVEN NO'S)

$$P_0(u) = 1 \qquad u = \cos\theta$$

$$P_0'(u) = 0$$

$$P_2(u) = \frac{1}{2}(3u^2 - 1)$$

$$P_2'(u) = \frac{1}{2}(6u)$$

$$P_4(u) = \frac{1}{8}(35u^4 - 30u^2 + 3)$$

$$P_4'(u) = \frac{1}{8}(140u^3 - 60u)$$

$$P_6(u) = \frac{1}{16}(231u^6 - 315u^4 + 105u^2 - 5)$$

$$P_6'(u) = \frac{1}{16}(1386u^5 - 1260u^3 + 210u)$$

$$P_8(u) = \frac{1}{128}(6{,}435u^8 - 12{,}012u^6 + 6{,}930u^4 - 1{,}260u^2 + 35)$$

$$P_8'(u) = \frac{1}{128}(51{,}480u^7 - 72{,}072u^5 + 27{,}720u^3 - 2{,}520)$$

SPACEING OF HELMHOLTZ PAIRS

TABLE 3

TABULATED VALUES OF K VS X AND A

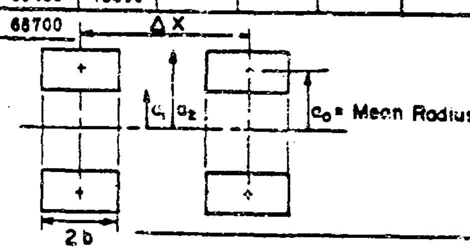
USED TO DETERMINE PROPER SPACING BETWEEN HELMHOLTZ COILS FOR NO 2ND DERIVATIVE AT THE ORIGIN. (UNIFORM CURRENT)

$$X = \frac{2b}{\frac{a_2 + a_1}{2}} = \frac{4\beta}{a+1}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1} = \frac{2(a-1)}{a+1}$$

$$K = \frac{\Delta X}{2a_0} = \frac{\Delta X}{a_1(1+a)}$$

A \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
0.0	.50000	.50160	.50600	51350	.52399	.53746	.55590	.57325	.59547	.62047	.64814	.67835	71092	.74569	.78246	.82192	.86118	.90275	94553
0.1	.49571	50022	50474	51227	52281	53535	55265	57228	59458	61966	64741	67770	71035	74519	78202	82064	86085	90246	94529
0.2	.49483	49636	50095	50860	51929	53300	54972	56938	59192	61724	64524	67576	70864	74369	78071	81951	85987	90161	94450
0.3	.48835	48992	49463	50246	51341	52744	54450	56455	58750	61324	64164	67256	70582	74121	77855	81763	85825	90021	94330
0.4	.47926	48089	48576	49337	50518	51965	53723	55783	58135	60768	63666	66814	70192	73780	77558	81505	85601	89827	94160
0.5	.46753	46924	47434	48261	49461	50968	52793	54925	57353	60062	63038	66255	69700	73350	77184	81181	85321	89586	94000
0.6	.45315	45495	46035	46929	48172	49754	51664	53883	56411	59215	62280	65587	69113	72838	76739	80796	84990	89300	
0.7	.43608	43801	44378	45332	46653	48329	50344	52680	55318	58235	61410	64820	68442	72254	76233	80359	84610	88970	
0.8	.41632	41840	42463	43491	44909	46701	48843	51313	54086	57136	60438	63967	67698	71608	75674	79877	84200	88600	
0.9	.39385	39613	40293	41412	42849	44880	47174	49802	52733	55936	59382	63044	66895	70915	75074	79360			
1.0	.36868	37120	37872	39014	40786	42884	45358	48169	51280	54655	58260	62068	66050	70183	74445	78800			
1.1	.34085	34368	35210	36580	38439	40738	43420	46442	49785	53319	57098	61061	65181	69446	73800				
1.2	.31043	31366	32321	33854	35937	38470	41397	44656	48191	51959	55922	60047	64309	68700					
1.3	.27755	28129	29229	30989	33321	[illegible]	39336	42656	46631	50613	54760	59050	63500						
1.4	.24239	24681	25969	28003	30652	33786	37296	41096	45120	49320									
1.5	.20518	21054	22597	24981	28009	[illegible]	35351	39440	43710										
1.6	.16622	17297	19195	22029	25505	29410	33590	38000											
1.7	.12585	13480	15690	19300	23300	27600	32100												
1.8	.08444	09690	12900	17000	21600														



AXIAL ERROR LIMITS

TABLE 4

TABULATION OF V VS X AND A

WHERE $V = z/a_0$ AND WHERE Z REPRESENTS THE POINT AT WHICH THE FIELD DIFFERS FROM THE CENTRAL FIELD BY 0.1 % (UNIFORM CURRENT)

$$X = \frac{2b}{\frac{a_2 + a_1}{2}} = \frac{4\beta}{\alpha + 1}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1} = \frac{2(\alpha - 1)}{\alpha + 1}$$

A \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
0.0	.172	.172	.173	.176	.179	.183	.188	.193	.200	.207	.216	.224	.234	.244	.255	.266	.278	.290	.302
0.1	.171	172	173	175	178	182	187	193	200	207	215	224	234	244	254	266	277	289	
0.2	.170	171	172	174	177	181	186	192	199	206	214	223	233	243	253	265	276	288	
0.3	.168	169	170	172	176	180	185	190	197	204	212	221	231	241	252	263	274	286	
0.4	.166	166	167	170	173	177	182	188	195	202	210	219	228	238	249	260	271		
0.5	.162	163	164	166	170	174	179	185	192	199	207	216	225	235	246	257	268		
0.6	.158	158	160	162	166	170	175	181	188	195	203	212	222	231	242	252			
0.7	.153	153	155	157	161	165	170	177	183	191	199	208	217	227	237	247			
0.8	.147	148	149	152	155	160	165	171	178	186	194	203	212	221	231				
0.9	.140	141	143	145	149	154	159	165	172	180	188	197	206	215	225				
1.0	.133	133	135	138	142	147	152	159	166	173	181	190	199	208					
1.1	.124	125	127	130	134	139	145	152	159	166	174	182	191						
1.2	.115	116	118	121	126	131	137	144	151	158	166	174							
1.3	.104	105	108	111	116	122	128	135	142	149									
1.4	.093	094	097	101	106	112	119	125	132										
1.5	.081	082	085	090	096														
1.6	.067																		

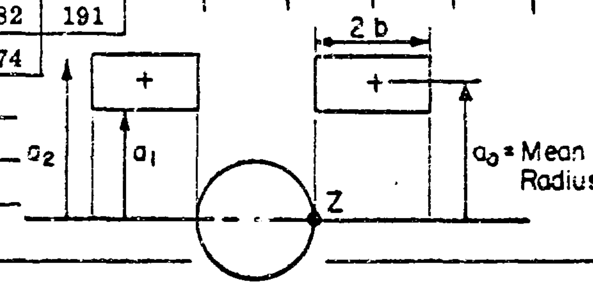


TABLE 5

AXIAL DISPLACEMENT ERROR

COEFFICIENT Q FOR IMPROPEPLY SPACED

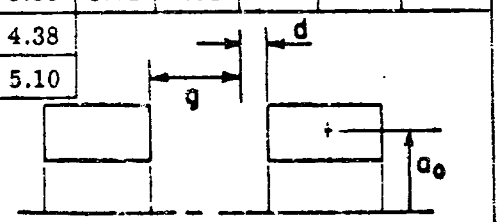
HELMHOLTZ COILS (UNIFORM CURRENT)

$$H = H_0\left(1 + Q\left(\frac{d}{a_0}\right)\left(\frac{z}{a_0}\right)^2 + \mathcal{E}_4\left(\frac{z}{a_0}\right)^4 + \cdots\right)$$

$$X = \frac{4b}{(a_2 + a_1)}$$

$$A = \frac{2(a_2 - a_1)}{a_2 + a_1}$$

A \ X	0.0 +	0.1 +	0.2 +	0.3 +	0.4 +	0.5 +	0.6 +	0.7 +	0.8 +	0.9 +	1.0 +	1.1 +	1.2 +	1.3 +	1.4 +	1.5 +	1.6 +	1.7 +
0.0	3.84	3.83	3.77	3.65	3.56	3.42	3.27	3.11	2.95	2.79	2.64	2.51	2.38	2.27	2.17	2.09	2.02	1.95
0.1	3.86	3.84	3.79	3.69	3.57	3.43	3.28	3.12	2.96	2.80	2.65	2.52	2.39	2.28	2.18	2.10	2.03	1.96
0.2	3.90	3.88	3.82	3.73	3.61	3.47	3.31	3.15	2.99	2.83	2.68	2.55	2.42	2.31	2.21	2.13	2.05	1.99
0.3	3.98	3.96	3.90	3.80	3.68	3.54	3.38	3.21	3.05	2.89	2.74	2.60	2.47	2.36	2.26	2.17	2.10	2.03
0.4	4.09	4.07	4.01	3.91	3.78	3.63	3.47	3.30	3.13	2.97	2.81	2.67	2.54	2.43	2.33	2.24	2.16	2.09
0.5	4.24	4.22	4.16	4.06	3.92	3.77	3.60	3.42	3.24	3.07	2.92	2.77	2.64	2.52	2.42	2.33	2.26	
0.6	4.46	4.44	4.37	4.26	4.11	3.95	3.77	3.58	3.40	3.22	3.06	2.91	2.77	2.65	2.55	2.46	2.37	
0.7	4.75	4.72	4.64	4.52	4.36	4.18	3.99	3.79	3.59	3.41	3.24	3.08	2.94	2.82	2.71	2.62		
0.8	5.13	5.10	5.01	4.87	4.70	4.50	4.28	4.07	3.86	3.66	3.48	3.31	3.17	3.04	2.93			
0.9	5.64	5.61	5.51	5.35	5.14	4.91	4.67	4.43	4.20	3.99	3.79	3.62	3.46	3.33	3.21			
1.0	6.37	6.32	6.19	5.99	5.75	5.47	5.19	4.92	4.66	4.42	4.21	4.02	3.85	3.71	3.61			
1.1	7.38	7.32	7.15	6.89	6.58	6.24	5.91	5.58	5.29	5.02	4.78	4.57	4.38					
1.2	8.87	8.79	8.55	8.19	7.77	7.34	6.91	6.51	6.16	5.84	5.57	5.33	5.10					
1.3	11.2	11.1	10.7	10.1	9.54	8.93	8.37	7.87	7.43	7.05								
1.4	15.0	14.8	14.1	13.2	12.3	11.4	10.6	9.93	9.36									
1.5	21.9	21.4	20.1	18.4	16.8													



g = Proper Spacing

d = Displacement Error

a_0 = Mean Radius

TABLE 6

FOURTH ORDER ERROR

COEFFICIENTS $\mathcal{E}_4$ FOR PROPERLY SPACED HELMHOLTZ COILS. (UNIFORM CURRENT)

$$H = H_0\left(1+\mathcal{E}_4\left(\frac{z}{a_0}\right)^4+\cdots\right)$$

$$X = \frac{4b}{a_2+a_1}$$

$$A = \frac{2(a_2-a_1)}{a_2+a_1}$$

A \ X	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	1.15	1.14	1.10	1.05	.978	.895	.805	.714	.625	.541	.464	.395	.334	.282	.237	.200	.168	.142
0.1	1.16	1.15	1.11	1.06	.986	.902	.811	.719	.629	.544	.466	.397	.336	.284	.239	.201	.169	.142
0.2	1.19	1.18	1.14	1.09	1.01	.923	.830	.735	.642	.555	.475	.404	.342	.288	.243	.204	.172	.145
0.3	1.25	1.23	1.20	1.13	1.05	.961	.862	.762	.664	.573	.490	.417	.352	.297	.250	.210	.177	.149
0.4	1.33	1.32	1.27	1.20	1.12	1.02	.909	.802	.698	.601	.513	.435	.367	.309	.260	.219	.184	.153
0.5	1.45	1.43	1.38	1.30	1.21	1.09	.976	.857	.743	.638	.544	.460	.388	.326	.274	.231	.194	
0.6	1.61	1.59	1.53	1.44	1.33	1.20	1.06	.931	.805	.688	.584	.493	.415	.349	.295	.246	.207	
0.7	1.83	1.81	1.74	1.63	1.49	1.34	1.18	1.03	.885	.754	.638	.537	.451	.379	.318	.267		
0.8	2.14	2.11	2.02	1.89	1.72	1.53	1.34	1.16	.991	.840	.708	.594	.498	.417	.350			
0.9	2.58	2.54	2.42	2.25	2.03	1.79	1.56	1.33	1.13	.954	.800	.669	.560	.468	.392			
1.0	3.23	3.17	3.00	2.76	2.46	2.15	1.85	1.57	1.32	1.11	.922	.769	.641	.536	.452			
1.1	4.20	4.11	3.87	3.51	3.09	2.66	2.26	1.89	1.58	1.31	1.09	.904	.753					
1.2	5.76	5.62	5.22	4.66	4.03	3.41	2.84	2.35	1.94	1.60	1.32	1.09	.907					
1.3	8.42	8.16	7.45	6.49	5.48	4.53	3.71	3.02	2.47	2.02								
1.4	13.4	12.8	11.4	9.62	7.85	6.31	5.05	4.05	3.27									
1.5	23.7	22.4	19.1	15.3	12.0													

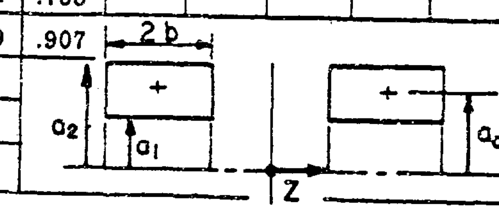


TABLE 7

SIXTH ORDER ERROR

COEFFICIENTS $\mathcal{E}_6$ FOR PROPERLY SPACED

HELMHOLTZ COILS (UNIFORM CURRENT)

$$H = H_o\left(1 + \cdots + \mathcal{E}_6\left(\frac{z}{a_o}\right)^6 + \cdots\right)$$

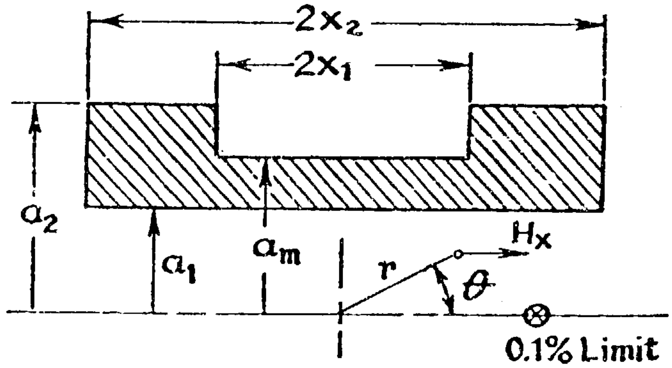
A \ X	0.0 +	0.1 +	0.2 +	0.3 +	0.4 +	0.5 +	0.6 +	0.7 +	0.8 +	0.9 +	1.0 +	1.1 +	1.2 +	1.3 +	1.4 +	1.5 +	1.6 +	1.7 +
0.0	1.26	1.25	1.20	1.13	1.04	.935	.828	.722	.621	.530	.449	.379	.318	.268	.225	.189	.160	.135
0.1	1.27	1.26	1.21	1.14	1.05	.946	.838	.731	.630	.538	.456	.384	.323	.272	.228	.192	.162	.137
0.2	1.31	1.30	1.25	1.18	1.09	.982	.871	.762	.658	.563	.478	.403	.340	.286	.241	.203	.172	.145
0.3	1.38	1.37	1.32	1.25	1.15	1.04	.930	.816	.707	.607	.517	.438	.369	.311	.262	.221	.187	.159
0.4	1.49	1.47	1.43	1.35	1.25	1.14	1.02	.901	.784	.675	.577	.490	.415	.351	.296	.250	.212	.181
0.5	1.64	1.63	1.58	1.50	1.40	1.28	1.16	1.03	.898	.777	.666	.568	.482	.408	.345	.292	.247	
0.6	1.86	1.84	1.80	1.72	1.62	1.49	1.36	1.21	1.06	.925	.796	.680	.579	.490	.415	.352	.299	
0.7	2.16	2.15	2.11	2.04	1.93	1.80	1.65	1.48	1.31	1.14	.987	.845	.720	.611	.518	.439		
0.8	2.60	2.59	2.57	2.51	2.51	2.27	2.10	1.89	1.68	1.47	1.27	1.09	.929	.789	.670			
0.9	3.25	3.26	3.26	3.24	3.15	3.01	2.79	2.54	2.26	1.98	1.71	1.46	1.25	1.06	.898			
1.0	4.25	4.29	4.37	4.42	4.38	4.22	3.94	3.58	3.18	2.78	2.40	2.05	1.75	1.48	1.22			
1.1	5.90	6.01	6.26	6.49	6.53	6.33	5.92	5.36	4.75	4.13	3.54	3.02	2.57					
1.2	8.77	9.08	9.79	10.4	10.6	10.3	9.58	8.60	7.54	6.50	5.55	4.71	4.00					
1.3	14.4	15.2	17.1	18.8	19.2	18.4	16.9	14.9	12.9	11.0								
1.4	26.8	29.4	35.0	39.0	39.4	36.9	32.8	28.4	24.1									
1.5	60.0	60.1	87.6	97.0	94.2													

TABLE 8

Typical 6th order solenoids (No 2nd or 4th derivative on axis) where length is fixed. (W. Garrett)

a_1	x_1/a_2	x_2/a_2	$\mathcal{E}_6 \times 10^{-3}$	$\mathcal{E}_8 \times 10^{-3}$	0.1 % Limits*
1.0	1.39327	1.82234	-5.504	-2.634	75
0.9	1.41768	1.82451	-4.918	-2.566	77
0.8	1.44747	1.82933	-4.255	-2.403	79
0.7	1.48310	1.83709	-3.551	-2.151	81
0.6	1.52472	1.84784	-2.849	-1.831	84
0.5	1.57193	1.86124	-2.194	-1.478	88

* % of a_2 along axis



$$a_2 \equiv 1.0$$

$$a_m \equiv \frac{a_2 + a_1}{2}$$

$$u = \cos\theta$$

$$H_x = H_0\left[1 + \mathcal{E}_6\left(\frac{r}{a_2}\right)^6 \rho_6(u) + \mathcal{E}_8\left(\frac{r}{a_2}\right)^8 \rho_8(u) + \cdots\right]$$

TABLE 9

Sixth Order Long Solenoids: $a_1 = 6$ cm, $a_2 = 12$ cm, $b = 36$ cm

Cancelled Second & Fourth Derivatives $(H = H_0(1+\mathcal{E}_2 z^2 + \mathcal{E}_4 z^4)$
$\mathcal{E}_2 = .607\times10^{-2}\,\%$, $\mathcal{E}_4 = .640\times10^{-5}\,\%$

	$i \alpha \frac{1}{\gamma}$: $a_1 = 6$cm, $a_2 = 12$cm, $b = 36$cm	Current sheet on I.D.	Current sheet on O.D.	One notch on O.D.	Uniformly reduced current in central region	Two separate sheets placed at ends on I.D.
Change in field at center of solenoid on axis in % of main field (−): decrease in field (+): increase in field	—	$\frac{I}{I'} = -207$ (−)0.45%	$\frac{I}{I'} = -41$ (−)2.76 %	$\frac{I}{I'} = -1$ (−)2.94 %	$\frac{I}{I'} = -71.5$ (−)1.01 %	$\frac{I}{I'} = +207$ (+)0.27 %
6th order error due to correction coil in % of main field at center	$(.536\times10^{-9})z^6(\%)$	$(.758\times10^{-6})z^6(\%)$	$(.303\times10^{-8})z^6(\%)$	$(.484\times10^{-6})z^6(\%)$	$(1.84\times10^{-4})z^6(\%)$	$(.758\times10^{-6})z^6(\%)$

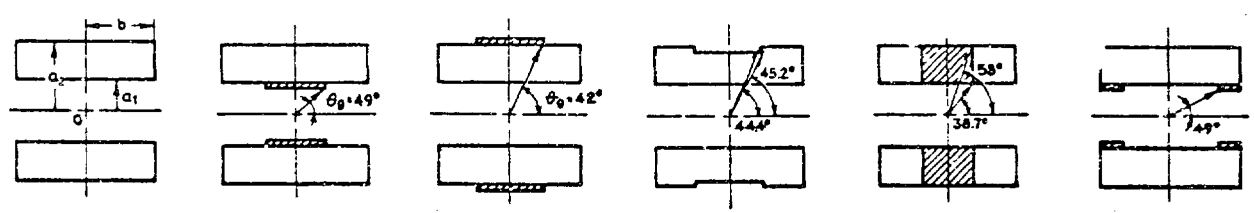


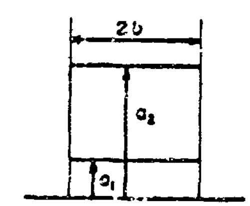
TABLE 10A

ERROR COEFFICIENTS AND GEOMETRY FACTORS FOR UNIFORM CURRENT DENSITY VS α AND β.

$$H = H_e\left(1 + D_2\left(\frac{z}{a_1}\right)^2 P_2(u) + D_4\left(\frac{z}{a_1}\right)^4 P_4(u) + \cdots\right)$$

NOTE: E-03 EXPONENT = 10^{-3}

ALPHA	BETA	C2	D4	D6	D8	G1	G102	G104	G106	G108
2	.5	-0.6058E 00	0.2612E-00	-0.6392E-01	-0.1640E-01	0.1333E-00	-0.8086E-01	0.3480E-01	-0.8518E-02	-0.2185E-02
3	.5	-0.4660E-00	0.1793E-00	-0.4264E-01	-0.9738E-02	0.1314E-00	-0.6121E-01	0.2356E-01	-0.5602E-02	-0.1279E-02
4	.5	-0.3924E-00	0.1435E-00	-0.3378E-01	-0.7637 02	0.1220E-00	-0.4786E-01	0.1752E-01	-0.4120E-02	-0.9316E-03
5	.5	-0.3469E-00	0.1238E-00	-0.2899E-01	-0.6545 02	0.1125E-00	-0.3902E-01	0.1352E-01	-0.3262E-02	-0.7363E-03
2	1	-0.3219E-00	0.2027E-01	0.1713E-01	-0.2553E-02	0.1627E-00	-0.5239E-01	0.3299E-02	0.2788E-02	-0.4156E-03
3	1	-0.2669E-00	0.2096E-01	0.9748E-02	-0.1584E-02	0.1661E-00	-0.4433E-01	0.3482E-02	0.1619E-02	-0.2631E-03
4	1	-0.2306E-00	0.1815E-01	0.7402E-02	-0.1218E-02	0.1571E-00	-0.3621E-01	0.2051E-02	0.1163E-02	-0.1912E-03
5	1	-0.2059E-00	0.1595E-01	0.6250E-02	-0.1031E-02	0.1464E-00	-0.3015E-01	0.2336E-02	0.9153E-03	-0.1510E-03
2	2	-0.8250E-01	-0.8532E-02	0.1017E-03	0.1396E-03	0.1638E-00	-0.1351E-01	-0.1398E-02	0.1665E-04	0.2286E-04
3	2	-0.8524E-01	-0.4561E-02	0.2520E-03	0.7333E-04	0.1789E-00	-0.1525E-01	-0.8157E-03	0.4508E-04	0.1312E-04
4	2	-0.8132E-01	-0.2807E-02	0.1986E-03	0.5220E-04	0.1762E-00	-0.1433E-01	-0.4946E-03	0.3500E-04	0.9196E-05
5	2	-0.7622E-01	-0.2028E-02	0.1601E-03	0.4291E-04	0.1687E-00	-0.1286E-01	-0.3423E-03	0.2701E-04	0.7240E-05
2	3	-0.2595E-01	-0.2402E-02	-0.1247E-03	-0.8041E-06	0.1492E-00	-0.3873E-02	-0.3585E-03	-0.1861E-04	-0.1200E-06
3	3	-0.3229E-01	-0.1886E-02	-0.4377E-04	0.1925E-05	0.1701E-00	-0.5491E-02	-0.3206E-03	-0.7443E-05	0.3273E-06
4	3	-0.3461E-01	-0.1360E-02	-0.1808E-04	0.1298E-05	0.1729E-00	-0.5983E-02	-0.2352E-03	-0.3126E-05	0.2244E-06
5	3	-0.3479E-01	-0.1017E-02	-0.1175E-04	0.9240E-06	0.1655E-00	-0.5897E-02	-0.1725E-03	-0.1991E-05	0.1566E-06
2	4	-0.1005E-01	-0.6920E-03	-0.3311E-04	-0.1074E-05	0.1353E-00	-0.1360E-02	-0.9364E-04	-0.4481E-05	-0.1453E-06
3	4	-0.1415E-01	-0.6995E-03	-0.2056E-04	-0.2459E-06	0.1580E-00	-0.2235E-02	-0.1105E-03	-0.3248E-05	-0.3885E-07
4	4	-0.1673E-01	-0.5918E-03	-0.1121E-04	-0.5189E-07	0.1641E-00	-0.2745E-02	-0.9712E-04	-0.1839E-05	-0.8515E-08
5	4	-0.1804E-01	-0.4755E-03	-0.6949E-05	-0.4398E-07	0.1638E-00	-0.2954E-02	-0.7792E-04	-0.1138E-05	-0.7202E-08
2	5	-0.4566E-02	-0.2315E-03	-0.8796E-05	-0.2648E-06	0.1238E-00	-0.5654E-03	-0.2867E-04	-0.1089E-05	-0.3279E-07
3	5	-0.6951E-02	-0.2766E-03	-0.7444E-05	-0.1329E-06	0.1467E-00	-0.1020E-02	-0.4058E-04	-0.1092E-05	-0.1950E-07
4	5	-0.8848E-02	-0.2666E-03	-0.5007E-05	-0.5276E-07	0.1546E-00	-0.1368E-02	-0.4152E-04	-0.7739E-06	-0.8155E-08
5	5	-0.1014E-01	-0.2365E-03	-0.3253E-05	-0.2886E-07	0.1562E-00	-0.1584E-02	-0.3694E-04	-0.5082E-06	-0.4508E-08
2	6	-0.2337E-02	-0.8914E-04	-0.2644E-05	-0.6574E-07	0.1145E-00	-0.2676E-03	-0.1021E-04	-0.3028E-06	-0.7529E-08
3	6	-0.3744E-02	-0.1190E-03	-0.2733E-05	-0.4747E-07	0.1369E-00	-0.5125E-03	-0.1629E-04	-0.3741E-06	-0.6498E-08
4	6	-0.5027E-02	-0.1287E-03	-0.2193E-05	-0.2516E-07	0.1456E-00	-0.7320E-03	-0.1874E-04	-0.3194E-06	-0.3664E-08
5	6	-0.6045E-02	-0.1234E-03	-0.1576E-05	-0.1328E-07	0.1485E-00	-0.8980E-03	-0.1833E-04	-0.2342E-06	-0.1973E-08
2	7	-0.1309E-02	-0.3855E-04	-0.9018E-06	-0.1819E-07	0.1069E-00	-0.1399E-03	-0.4121E-05	-0.9640E-07	-0.1945E-08
3	7	-0.2171E-02	-0.5553E-04	-0.1067E-05	-0.1650E-07	0.1285E-00	-0.2790E-03	-0.7136E-05	-0.1372E-06	-0.2121E-08
4	7	-0.3030E-02	-0.6508E-04	-0.9330E-06	-0.1093E-07	0.1376E-00	-0.4169E-03	-0.8955E-05	-0.1353E-06	-0.1504E-08
5	7	-0.3784E-02	-0.6715E-04	-0.7865E-06	-0.6362E-08	0.1413E-00	-0.5347E-03	-0.9489E-05	-0.1111E-06	-0.8993E-09
2	8	-0.7843E-03	-0.1832E-04	-0.3437E-06	-0.5654E-08	0.1005E-00	-0.7904E-04	-0.1842E-05	-0.3455E-07	-0.5684E-09
3	8	-0.1336E-02	-0.2784E-04	-0.4473E-06	-0.5987E-08	0.1213E-00	-0.1621E-03	-0.3378E-05	-0.5428E-07	-0.7265E-09
4	8	-0.1918E-02	-0.3467E-04	-0.4576E-06	-0.4698E-08	0.1305E-00	-0.2504E-03	-0.4525E-05	-0.5973E-07	-0.6132E-09
5	8	-0.2467E-02	-0.3795E-04	-0.4027E-06	-0.3106E-08	0.1347E-00	-0.3323E-03	-0.5113E-05	-0.5425E-07	-0.4184E-09
2	9	-0.4993E-03	-0.9406E-05	-0.1440E-06	-0.1952E-08	0.9512E-01	-0.4750E-04	-0.8948E-06	-0.1369E-07	-0.1857E-09
3	9	-0.8630E-03	-0.1485E-04	-0.2003E-06	-0.2308E-08	0.1152E-00	-0.9938E-04	-0.1710E-05	-0.2310E-07	-0.2658E-09
4	9	-0.1266E-02	-0.1937E-04	-0.2224E-06	-0.2060E-08	0.1243E-00	-0.1574E-03	-0.2407E-05	-0.2764E-07	-0.2561E-09
5	9	-0.1666E-02	-0.2223E-04	-0.2119E-06	-0.1533E-08	0.1288E-00	-0.2145E-03	-0.2863E-05	-0.2728E-07	-0.1974E-09
2	10	-0.3317E-03	-0.5143E-05	-0.6526E-07	-0.7388E-09	0.9048E-01	-0.3001E-04	-0.4655E-06	-0.5905E-08	-0.6685E-10
3	10	-0.5806E-03	-0.8360E-05	-0.9568E-07	-0.9474E-09	0.1098E-00	-0.6373E-04	-0.9176E-06	-0.1050E-07	-0.1040E-09
4	10	-0.8636E-03	-0.1129E-04	-0.1129E-06	-0.9337E-09	0.1188E-00	-0.1028E-03	-0.1342E-05	-0.1541E-07	-0.1109E-09
5	10	-0.1160E-02	-0.1347E-04	-0.1147E-06	-0.7682E-09	0.1234E-00	-0.1431E-03	-0.1662E-05	-0.1416E-07	-0.9480E-10



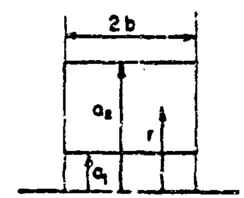
$\alpha = a_2/a_1$
$\beta = b/a_1$
j = Constant

TABLE 10B

ERROR COEFFICIENTS AND GEOMETRY FACTORS FOR RADIAL CURRENT DENSITY $j = \left(\frac{j_1 a_1}{r}\right)$ VS. α AND β

$$H = H_0 \left(1 + A_2\left(\frac{z}{a_1}\right)^2 P_2(u) + A\left(\frac{z}{a_1}\right)^4 P_4(u) + \cdots\right)$$

ALPHA	BETA	A2	A4	A6	A8	G2	G2A2	G2A4	G2A6	G2A8
2	.5	-0.6432E 00	0.2867E-00	-0.6671E-01	0.4245E-00	0.1408E-00	-0.9053E-01	0.4035E-01	-0.9671E-02	0.5975E-01
3	.5	-0.5392E 00	0.2248E-00	-0.5307E-01	0.3167E-00	0.1508E-00	-0.8131E-01	0.3339E-01	-0.8004E-02	0.4776E-01
4	.5	-0.4910E-00	0.2002E-00	-0.4707E-01	0.2801E-00	0.1518E-00	-0.7454E-01	0.3039E-01	-0.7146E-02	0.4253E-01
5	.5	-0.4639E-00	0.1875E-00	-0.4402E-01	0.2619E-00	0.1507E-00	-0.6991E-01	0.2825E-01	-0.6635E-02	0.3947E-01
2	1	-0.3300E-00	0.1643E-01	0.2021E-01	0.8493E-01	0.1734E-00	-0.5623E-01	0.2800E-02	0.3444E-02	0.1447E-01
3	1	-0.2906E-00	0.1816E-01	0.1423E-01	0.6202E-01	0.1873E-00	-0.5444E-01	0.3402E-02	0.2666E-02	0.1162E-01
4	1	-0.2676E-00	0.1698E-01	0.1236E-01	0.5422E-01	0.1909E-00	-0.5108E-01	0.3241E-02	0.2360E-02	0.1035E-01
5	1	-0.2534E-00	0.1603E-01	0.1147E-01	0.5035E-01	0.1908E-00	-0.4834E-01	0.3059E-02	0.2188E-02	0.9605E-02
2	2	-0.8048E-01	-0.9132E-02	-0.3720E-04	0.1947E-02	0.1693E-00	-0.1362E-01	-0.1554E-02	-0.6297E-05	0.3296E-03
3	2	-0.8321E-01	-0.6197E-02	0.1202E-03	0.1427E-02	0.1957E-00	-0.1629E-01	-0.1213E-02	0.2352E-04	0.2911E-03
4	2	-0.8132E-01	-0.4937E-02	0.1096E-03	0.1275E-02	0.2049E-00	-0.1666E-01	-0.1012E-02	0.2245E-04	0.2613E-03
5	2	-0.7881E-01	-0.4366E-02	0.9816E-04	0.1166E-02	0.2082E-00	-0.1641E-01	-0.9088E-03	0.2044E-04	0.2428E-03
2	3	-0.2474E-01	-0.2397E-02	-0.1373E-03	0.7170E-04	0.1533E-00	-0.3794E-02	-0.3674E-03	-0.2106E-04	0.1099E-04
3	3	-0.2966E-01	-0.2044E-02	-0.7536E-04	0.6974E-04	0.1830E-00	-0.5427E-02	-0.3740E-03	-0.1379E-04	0.1276E-04
4	3	-0.3149E-01	-0.1709E-02	-0.5466E-04	0.6185E-04	0.1956E-00	-0.6159E-02	-0.3342E-03	-0.1069E-04	0.1210E-04
5	3	-0.3190E-01	-0.1498E-02	-0.4773E-04	0.5631E-04	0.2016E-00	-0.6432E-02	-0.3020E-03	-0.9622E-05	0.1135E-04
2	4	-0.9481E-02	-0.6701E-03	-0.3354E-04	0.4533E-05	0.1386E-00	-0.1314E-02	-0.9289E-04	-0.4650E-05	0.6283E-06
3	4	-0.1258E-01	-0.6865E-03	-0.2461E-04	0.5984E-05	0.1684E-00	-0.2119E-02	-0.1156E-03	-0.4146E-05	0.1008E-05
4	4	-0.1441E-01	-0.6255E-03	-0.1827E-04	0.5956E-05	0.1827E-00	-0.2632E-02	-0.1143E-03	-0.3337E-05	0.1088E-05
5	4	-0.1536E-01	-0.5613E-03	-0.1525E-04	0.5582E-05	0.1903E-00	-0.2922E-02	-0.1068E-03	-0.2902E-05	0.1062E-05
2	5	-0.4283E-02	-0.2209E-03	-0.8620E-05	0.4312E-06	0.1267E-00	-0.5425E-03	-0.2798E-04	-0.1092E-05	0.5461E-07
3	5	-0.6073E-02	-0.2577E-03	-0.7776E-05	0.7453E-06	0.1556E-00	-0.9447E-03	-0.4009E-04	-0.1210E-05	0.1160E-06
4	5	-0.7380E-02	-0.2569E-03	-0.6253E-05	0.8680E-06	0.1703E-00	-0.1257E-02	-0.4375E-04	-0.1065E-05	0.1478E-06
5	5	-0.8236E-02	-0.2414E-03	-0.5174E-05	0.8717E-06	0.1788E-00	-0.1473E-02	-0.4317E-04	-0.9253E-06	0.1559E-06
2	6	-0.2185E-02	-0.8434E-04	-0.2546E-05	0.5558E-07	0.1170E-00	-0.2557E-03	-0.9871E-05	-0.2980E-06	0.6505E-08
3	6	-0.3236E-02	-0.1076E-03	-0.2661E-05	0.1201E-06	0.1447E-00	-0.4683E-03	-0.1557E-04	-0.3851E-06	0.1738E-07
4	6	-0.4109E-02	-0.1258E-03	-0.2354E-05	0.1635E-06	0.1594E-00	-0.6550E-03	-0.1846E-04	-0.3752E-06	0.2607E-07
5	6	-0.4765E-02	-0.1147E-03	-0.2005E-05	0.1803E-06	0.1684E-00	-0.8023E-03	-0.1930E-04	-0.3375E-06	0.3035E-07
2	7	-0.1221E-02	-0.3629E-04	-0.8598E-06	0.8960E-08	0.1092E-00	-0.1333E-03	-0.3962E-05	-0.9387E-07	0.9783E-09
3	7	-0.1863E-02	-0.4925E-04	-0.9976E-06	0.2350E-07	0.1356E-00	-0.2526E-03	-0.6677E-05	-0.1353E-06	0.3187E-08
4	7	-0.2443E-02	-0.5635E-04	-0.9626E-06	0.3694E-07	0.1500E-00	-0.3665E-03	-0.8453E-05	-0.1444E-06	0.5526E-08
5	7	-0.2922E-02	-0.5861E-04	-0.8600E-06	0.4483E-07	0.1591E-00	-0.4648E-03	-0.9323E-05	-0.1368E-06	0.7131E-08
2	8	-0.7327E-03	-0.1719E-04	-0.3256E-06	0.1694E-08	0.1026E-00	-0.7519E-04	-0.1764E-05	-0.3342E-07	0.1738E-09
3	8	-0.1141E-02	-0.2439E-04	-0.4074E-06	0.5365E-08	0.1276E-00	-0.1458E-03	-0.3117E-05	-0.5207E-07	0.6859E-09
4	8	-0.1533E-02	-0.2925E-04	-0.4226E-06	0.9515E-08	0.1419E-00	-0.2174E-03	-0.4150E-05	-0.5995E-07	0.1350E-08
5	8	-0.1878E-02	-0.3174E-04	-0.3978E-06	0.1272E-07	0.1509E-00	-0.2834E-03	-0.4790E-05	-0.6003E-07	0.1919E-08
2	9	-0.4648E-03	-0.8803E-05	-0.1357E-06	0.3539E-09	0.9709E-01	-0.4513E-04	-0.8567E-06	-0.1318E-07	0.3436E-10
3	9	-0.7347E-03	-0.1290E-04	-0.1795E-06	0.1385E-08	0.1212E-00	-0.8904E-04	-0.1563E-05	-0.2176E-07	0.1678E-09
4	9	-0.1005E-02	-0.1605E-04	-0.1974E-06	0.2745E-08	0.1348E-00	-0.1355E-03	-0.2164E-05	-0.2661E-07	0.3701E-09
5	9	-0.1255E-02	-0.1805E-04	-0.1952E-06	0.3997E-08	0.1437E-00	-0.1804E-03	-0.2593E-05	-0.2805E-07	0.5744E-09
2	10	-0.3085E-03	-0.4807E-05	-0.6134E-07	0.7605E-10	0.9234E-01	-0.2849E-04	-0.4439E-06	-0.5664E-08	0.7023E-11
3	10	-0.4931E-03	-0.7216E-05	-0.8458E-07	0.3940E-09	0.1154E-00	-0.5692E-04	-0.8329E-06	-0.9764E-08	0.4548E-10
4	10	-0.6838E-03	-0.9238E-05	-0.9745E-07	0.8679E-09	0.1286E-00	-0.8796E-04	-0.1188E-05	-0.1254E-07	0.1116E-09
5	10	-0.8673E-03	-0.1070E-04	-0.1006E-06	0.1366E-08	0.1374E-00	-0.1191E-03	-0.1469E-05	-0.1382E-07	0.1876E-09



$\alpha = a_2/a_1$

$\beta = b/a_1$

$j = \frac{j_1 a_1}{r}$

TABLE IIA

SECOND ERROR COEFFICIENTS AND GEOMETRY FACTORS FOR SMALL β's. $(0.1 < \beta < 1.0)$

UNIFORM CURRENT: $H = H_0\left(1 + D_2\left(\frac{z}{a_1}\right)^2 P_2(u) + \cdots\right)$

RADIAL CURRENT: $H = H_0\left(1 + A_2\left(\frac{z}{a_1}\right)^2 P_2(u) + \cdots\right)$

ALPHA	BETA	D2	A2	G1	G2	G1D2	G2A2
0.11E 01	0.20E-00	-0.127109E 01	-0.127288E 01	0.458034E-01	0.458542E-01	-0.582203E-01	-0.583668E-01
0.11E 01	0.40E-00	-0.103960E 01	-0.104036E 01	0.616104E-01	0.616748E-01	-0.640556E-01	-0.641946E-01
0.11E 01	0.60E 00	-0.773785E 00	-0.774379E 00	0.700961E-01	0.701627E-01	-0.542394E-01	-0.543325E-01
0.11E 01	0.80E 00	-0.544580E 00	-0.544798E 00	0.741416E-01	0.742052E-01	-0.403760E-01	-0.404268E-01
0.11E 01	1.00E 00	-0.373856E-00	-0.373803E-00	0.754555E-01	0.755141E-01	-0.282095E-01	-0.282334E-01
0.30E 01	0.20E-00	-0.577935E 00	-0.686047E 00	0.863301E-01	0.998803E-01	-0.500411E-01	-0.685223E-01
0.30E 01	0.40E-00	-0.526992E 00	-0.594620E 00	0.119413E-00	0.137482E-00	-0.607802E-01	-0.817465E-01
0.30E 01	0.60E 00	-0.421749E-00	-0.482735E-00	0.141226E-00	0.161610E-00	-0.595619E-01	-0.780144E-01
0.30E 01	0.80E 00	-0.337591E-00	-0.377542E-00	0.156097E-00	0.177366E-00	-0.527597E-01	-0.669631E-01
0.30E 01	1.00E 00	-0.266920E-00	-0.290591E-00	0.166092E-00	0.187341E-00	-0.443345E-01	-0.544396E-01
0.50E 01	0.20E-00	-0.427650E-00	-0.589087E 00	0.732226E-01	0.991589E-01	-0.313137E-01	-0.584132E-01
0.50E 01	0.40E-00	-0.377535E-00	-0.511084E 00	0.101821E-00	0.137015E-00	-0.384412E-01	-0.700261E-01
0.50E 01	0.60E 00	-0.315495E-00	-0.415941E-00	0.121531E-00	0.162012E-00	-0.383422E-01	-0.673873E-01
0.50E 01	0.80E 00	-0.256116E-00	-0.326811E-00	0.135873E-00	0.179117E-00	-0.347992E-01	-0.585375E-01
0.50E 01	1.00E 00	-0.208750E-00	-0.253419E-00	0.146445E-00	0.190760E-00	-0.301530E-01	-0.483423E-01

TABLE IIB

FOURTH ERROR COEFFICIENTS AND GEOMETRY FACTORS FOR SMALL β's. $(0.1 < \beta < 2.0)$

UNIFORM CURRENT: $H = H_0\left(1 + \cdots + D_4\left(\frac{z}{a_1}\right)^4 P_4(u) + \cdots\right)$

RADIAL CURRENT: $H = H_0\left(1 + \cdots + A_4\left(\frac{z}{a_1}\right)^4 P_4(u) + \cdots\right)$

ALPHA	BETA	D4	A4	G1	G2	C1D4	G2A4
0.11E 01	0.20E-00	0.128397E 01	0.128751E 01	0.458034E-01	0.458542E-01	0.583111E-01	0.590377E-01
0.11E 01	0.40E-00	0.726285E 00	0.727458E 00	0.616104E-01	0.616748E-01	0.447471E-01	0.448782E-01
0.11E 01	0.60E 00	0.250530E-00	0.250630E-00	0.700961E-01	0.701627E-01	0.196640E-01	0.196897E-01
0.11E 01	0.80E 00	0.549410E-01	0.546943E-01	0.741416E-01	0.742052E-01	0.407341E-02	0.405860E-02
0.11E 01	1.00E 00	-0.248591E-01	-0.250764E-01	0.754555E-01	0.755141E-01	-0.187576E-02	-0.189362E-02
0.11E 01	0.12E 01	-0.405711E-01	-0.406574E-01	0.751607E-01	0.752146E-01	-0.304936E-02	-0.306109E-02
0.11E 01	0.14E 01	-0.355153E-01	-0.355772E-01	0.735675E-01	0.740156E-01	-0.262697E-02	-0.263332E-02
0.11E 01	0.16E 01	-0.265260E-01	-0.265540E-01	0.723064E-01	0.723502E-01	-0.191805E-02	-0.192119E-02
0.11E 01	0.18E 01	-0.186610E-01	-0.186111E-01	0.704315E-01	0.704716E-01	-0.131573E-02	-0.131719E-02
0.11E 01	0.20E 01	-0.120249E-01	-0.120873E-01	0.684391E-01	0.685262E-01	-0.882477E-03	-0.883120E-03
0.30E 01	0.20E-00	0.355055E-00	0.479535E-00	0.863301E-01	0.998803E-01	0.314333E-01	0.478962E-01
0.30E 01	0.40E-00	0.240074E-00	0.307604E-00	0.119413E-00	0.137482E-00	0.286678E-01	0.423174E-01
0.30E 01	0.60E 00	0.127221E-00	0.155356E-00	0.141226E-00	0.161610E-00	0.180516E-01	0.251087E-01
0.30E 01	0.80E 00	0.572251E-01	0.623432E-01	0.156097E-00	0.177366E-00	0.393263E-02	0.111640E-01
0.30E 01	1.00E 00	0.209614E-01	0.181620E-01	0.166092E-00	0.187341E-00	0.348151E-02	0.340248E-02
0.30E 01	0.12E 01	0.452175E-02	-0.304825E-03	0.172561E-00	0.193252E-00	0.778552E-03	-0.589080E-04
0.30E 01	0.14E 01	-0.219366E-02	-0.660326E-02	0.176464E-00	0.196308E-00	-0.387101E-03	-0.129628E-02
0.30E 01	0.16E 01	-0.448744E-02	-0.787627E-02	0.178497E-00	0.197366E-00	-0.800993E-03	-0.155449E-02
0.30E 01	0.18E 01	-0.465471E-02	-0.730318E-02	0.179172E-00	0.197053E-00	-0.877347E-03	-0.143897E-02
0.30E 01	0.20E 01	-0.456065E-02	-0.619650E-02	0.178361E-00	0.195741E-00	-0.815721E-03	-0.121291E-02
0.50E 01	0.20E-00	0.250937E-00	0.400334E-00	0.732226E-01	0.991589E-01	0.183742E-01	0.397463E-01
0.50E 01	0.40E-00	0.165522E-00	0.256672E-00	0.101821E-00	0.137015E-00	0.168537E-01	0.351953E-01
0.50E 01	0.60E 00	0.455350E-01	0.129643E-00	0.121531E-00	0.162012E-00	0.107595E-01	0.210037E-01
0.50E 01	0.80E 00	0.404784E-01	0.529574E-01	0.135873E-00	0.179117E-00	0.549991E-02	0.948559E-02
0.50E 01	1.00E 00	0.159562E-01	0.160348E-01	0.146445E-00	0.190760E-00	0.233581E-02	0.305881E-02
0.50E 01	0.12E 01	0.421557E-02	0.873049E-03	0.154235E-00	0.198509E-00	0.743348E-03	0.173308E-03
0.50E 01	0.14E 01	0.187284E-03	-0.433023E-02	0.159924E-00	0.203455E-00	0.299521E-04	-0.881007E-03
0.50E 01	0.16E 01	-0.153932E-02	-0.546761E-02	0.164017E-00	0.206372E-00	-0.252474E-03	-0.112836E-02
0.50E 01	0.18E 01	-0.203179E-02	-0.512875E-02	0.166865E-00	0.207813E-00	-0.339034E-03	-0.106582E-02
0.50E 01	0.20E 01	-0.202846E-02	-0.436567E-02	0.168747E-00	0.208173E-00	-0.342295E-03	-0.908823E-03

TABLE IIC

SIXTH ERROR COEFFICIENTS .....

SMALL β's. $(0.1 < \beta < 3)$

UNIFORM CURRENT : $H = H_0\left(1 + \cdots + D_6\left(\frac{z}{a_1}\right)^6 P_6(u) + \cdots\right)$

RADIAL CURRENT : $H = H_0\left(1 + \cdots + A_6\left(\frac{z}{a_1}\right)^6 P_6(u) + \cdots\right)$

ALPHA	BETA	D6	A6	G1	G2	G1D6	G2A6
0.11E 01	0.20E-00	-0.114744E 01	-0.115192E 01	0.458034E-01	0.458542E-01	-0.525566E-01	-0.528202E-01
0.11E 01	0.40E-00	-0.327958E-00	-0.320502E-00	0.616108E-01	0.616748E-01	-0.202057E-01	-0.202603E-01
0.11E 01	0.60E 00	0.429624E-01	0.435002E-01	0.700961E-01	0.701627E-01	0.301150E-02	0.305209E-02
0.11E 01	0.80E 00	0.825501E-01	0.828536E-01	0.741416E-01	0.742052E-01	0.612099E-02	0.614816E-02
0.11E 01	1.00E 00	0.444103E-01	0.444703E-01	0.754556E-01	0.755141E-01	0.335101E-02	0.335814E-02
0.11E 01	0.12E 01	0.160882E-01	0.160742E-01	0.751609E-01	0.752140E-01	0.120920E-02	0.120901E-02
0.11E 01	0.14E 01	0.368124E-02	0.366027E-02	0.739675E-01	0.740156E-01	0.272292E-03	0.270917E-03
0.11E 01	0.16E 01	-0.433870E-03	-0.447750E-03	0.723064E-01	0.723502E-01	-0.313716E-04	-0.323948E-04
0.11E 01	0.18E 01	-0.134460E-02	-0.135209E-02	0.704315E-01	0.704716E-01	-0.947023E-04	-0.952843E-04
0.11E 01	0.20E 01	-0.125399E-02	-0.125769E-02	0.684891E-01	0.685262E-01	-0.858849E-04	-0.861845E-04
0.11E 01	0.22E 01	-0.949194E-03	-0.950915E-03	0.665616E-01	0.665961E-01	-0.631798E-04	-0.633273E-04
0.11E 01	0.24E 01	-0.666185E-03	-0.666947E-03	0.646936E-01	0.647259E-01	-0.430979E-04	-0.431688E-04

0.11E 01	0.26E 01	-0.453687E-03	-0.[illegible]52998E-03	0.629076E-01	0.629381E-01	-0.285403E-04	-0.285738E-04
0.11E 01	0.28E 01	-0.305942E-03	-0.306049E-03	0.612135E-01	0.612424E-01	-0.187278E-04	-0.187432E-04
0.11E 01	0.30E 01	-0.206305E-03	-0.206405E-03	0.596135E-01	0.596410E-01	-0.123033E-04	-0.123102E-04
0.30E 01	0.20E-00	-0.242637E-00	-0.340836E-00	0.863901E-01	0.998805E-01	-0.210478E-01	-0.340429E-01
0.30E 01	0.40E-00	-0.943680E-01	-0.125969E-00	0.119413E-00	0.137482E-00	-0.112687E-01	-0.173185E-01
0.30E 01	0.60E 00	-0.112737E-01	-0.100209E-01	0.141226E-00	0.161610E-00	-0.159215E-02	-0.161947E-02
0.30E 01	0.80E 00	0.103659E-01	0.172626E-01	0.156097E-00	0.177366E-00	0.161808E-02	0.306180E-02
0.30E 01	1.00E 00	0.974764E-02	0.142310E-01	0.166092E-00	0.187341E-00	0.161904E-02	0.266605E-02
0.30E 01	0.12E 01	0.577425E-02	0.776106E-02	0.172561E-00	0.193252E-00	0.996410E-03	0.149984E-02
0.30E 01	0.14E 01	0.294275E-02	0.359257E-02	0.176464E-00	0.196308E-00	0.519289E-03	0.705248E-03
0.30E 01	0.16E 01	0.139454E-02	0.148599E-02	0.178497E-00	0.197366E-00	0.248921E-03	0.293284E-03
0.30E 01	0.18E 01	0.622447E-03	0.527236E-03	0.179171E-00	0.197033E-00	0.111524E-03	0.103883E-03
0.30E 01	0.20E 01	0.252048E-03	0.120160E-03	0.178861E-00	0.195741E-00	0.450815E-04	0.235201E-04
0.30E 01	0.22E 01	0.787450E-04	-0.386301E-04	0.177843E-00	0.193794E-00	0.140040E-04	-0.748627E-05
0.30E 01	0.24E 01	0.527543E-06	-0.902879E-04	0.176320E-00	0.191409E-00	0.930165E-07	-0.172819E-04
0.30E 01	0.26E 01	-0.320600E-04	-0.979137E-04	0.174445E-00	0.188743E-00	-0.559270E-05	-0.184805E-04
0.30E 01	0.28E 01	-0.429428E-04	-0.890411E-04	0.172328E-00	0.185906E-00	-0.740024E-05	-0.165532E-04
0.30E 01	0.30E 01	-0.437708E-04	-0.753634E-04	0.170053E-00	0.182977E-00	-0.744337E-05	-0.137897E-04
0.50E 01	0.20E-00	-0.166245E-00	-0.283817E-00	0.732226E-01	0.991589E-01	-0.121729E-01	-0.281430E-01
0.50E 01	0.40E-00	-0.641582E-01	-0.104585E-00	0.101821E-00	0.137015E-00	-0.653269E-02	-0.143297E-01
0.50E 01	0.60E 00	-0.778815E-02	-0.839155E-02	0.121531E-00	0.162012E-00	-0.946502E-03	-0.135953E-02
0.50E 01	0.80E 00	0.669620E-02	0.140163E-01	0.135873E-00	0.179117E-00	0.909831E-03	0.251057E-02
0.50E 01	1.00E 00	0.625015E-02	0.114679E-01	0.146445E-00	0.190760E-00	0.915302E-03	0.218761E-02
0.50E 01	0.12E 01	0.363998E-02	0.618905E-02	0.154235E-00	0.198509E-00	0.561414E-03	0.122858E-02
0.50E 01	0.14E 01	0.182079E-02	0.283239E-02	0.159928E-00	0.203455E-00	0.291196E-03	0.576263E-03
0.50E 01	0.16E 01	0.850042E-03	0.115950E-02	0.164017E-00	0.206372E-00	0.139421E-03	0.239289E-03
0.50E 01	0.18E 01	0.379212E-03	0.410272E-03	0.166865E-00	0.207813E-00	0.632772E-04	0.852599E-04
0.50E 01	0.20E 01	0.160062E-03	0.981818E-04	0.168747E-00	0.208175E-00	0.270099E-04	0.204390E-04
0.50E 01	0.22E 01	0.604136E-04	-0.207735E-04	0.169870E-00	0.207746E-00	0.102625E-04	-0.431561E-05
0.50E 01	0.24E 01	0.162126E-04	-0.583425E-04	0.170396E-00	0.206737E-00	0.276256E-05	-0.120615E-04
0.50E 01	0.26E 01	-0.253418E-05	-0.635182E-04	0.170449E-00	0.205303E-00	-0.431948E-06	-0.130405E-04
0.50E 01	0.28E 01	-0.971601E-05	-0.571356E-04	0.170127E-00	0.203561E-00	-0.165295E-05	-0.116306E-04
0.50E 01	0.30E 01	-0.117469E-04	-0.477299E-04	0.169508E-00	0.201598E-00	-0.199119E-05	-0.962224E-05

TABLE II D

EIGHTH ERROR COEFFICIENT

SMALL β's (0.1 β 4.0)

UNIFORM CURRENT: $H = H_0\left(1 + \cdots + D_8\left(\frac{z}{a_1}\right)^8 P_8(u) + \cdots\right)$

RADIAL CURRENT: $H = H_0\left(1 + \cdots A_8\left(\frac{z}{a_1}\right)^8 P_8(u) + \cdots\right)$

ALPHA	BETA	C8	A8	G1	G2	G108	D8
0.11E 01	0.20E-00	0.925946E 00	0.566557E 01	0.458034E-01	0.458542E-01	0.424115E-01	0.259790E-00
0.11E 01	0.40E-00	0.151747E-01	0.173807E 01	0.616108E-01	0.616748E-01	0.934923E-03	0.107195E-00
0.11E 01	0.60E 00	-0.115020E-00	0.884654E 00	0.700961E-01	0.701627E-01	-0.506247E-02	0.620656E-01
0.11E 01	0.80E 00	-0.323291E-01	0.454440E-00	0.741416E-01	0.742052E-01	-0.239693E-02	0.338702E-01
0.11E 01	1.00E 00	0.285604E-02	0.206424E-00	0.754556E-01	0.755141E-01	0.218522E-03	0.157389E-01
0.11E 01	0.12E 01	0.597359E-02	0.865286E-01	0.751609E-01	0.752140E-01	0.449161E-03	0.650816E-02
0.11E 01	0.14E 01	0.325354E-02	0.343286E-01	0.739675E-01	0.740156E-01	0.234963E-03	0.254085E-02
0.11E 01	0.16E 01	0.125769E-02	0.135052E-01	0.723064E-01	0.723502E-01	0.909389E-04	0.937103E-03
0.11E 01	0.18E 01	0.397601E-03	0.538915E-02	0.704335E-01	0.704716E-01	0.280036E-04	0.379782E-03
0.11E 01	0.20E 01	0.854969E-04	0.220872E-02	0.684891E-01	0.685262E-01	0.585546E-05	0.151355E-03
0.11E 01	0.22E 01	-0.940588E-05	0.935458E-03	0.665816E-01	0.665961E-01	-0.626070E-06	0.622979E-04
0.11E 01	0.24E 01	-0.293119E-04	0.410379E-03	0.646934E-01	0.647259E-01	-0.189629E-05	0.265621E-04
0.11E 01	0.26E 01	-0.270376E-04	0.186501E-03	0.629076E-01	0.629381E-01	-0.170087E-05	0.117380E-04
0.11E 01	0.28E 01	-0.202367E-04	0.877083E-04	0.612135E-01	0.612424E-01	-0.123876E-05	0.537149E-05
0.11E 01	0.30E 01	-0.140228E-04	0.426103E-04	0.596135E-01	0.596410E-01	-0.835948E-06	0.254132E-05
0.11E 01	0.32E 01	-0.940702E-05	0.213409E-04	0.581058E-01	0.581321E-01	-0.546603E-06	0.124059E-05
0.11E 01	0.34E 01	-0.623280E-05	0.109953E-04	0.567865E-01	0.567116E-01	-0.353316E-06	0.623561E-06
0.11E 01	0.34E 01	-0.412080E-05	0.581514E-05	0.553504E-01	0.553744E-01	-0.228088E-06	0.322011E-06
0.11E 01	0.38E 01	-0.273397E-05	0.315047E-05	0.540919E-01	0.541153E-01	-0.147885E-06	0.170489E-06
0.11E 01	0.40E 01	-0.182600E-05	0.174497E-05	0.529057E-01	0.529282E-01	-0.966058E-07	0.923580E-07
0.30E 01	0.20E-00	0.164182E-00	0.154838E 01	0.863901E-01	0.998805E-01	0.141837E-01	0.154653E-00
0.30E 01	0.40E-00	0.180170E-01	0.460407E-00	0.119413E-00	0.137482E-00	0.215146E-02	0.632974E-01
0.30E 01	0.60E 00	-0.160899E-01	0.230263E-00	0.141226E-00	0.161610E-00	-0.227232E-02	0.372127E-01
0.30E 01	0.80E 00	-0.803785E-02	0.123600E-00	0.156097E-00	0.177366E-00	-0.122468E-02	0.219224E-01
0.30E 01	1.00E 00	-0.158398E-02	0.620185E-01	0.166092E-00	0.187341E-00	-0.263087E-03	0.116186E-01
0.30E 01	0.12E 01	0.232466E-03	0.293510E-01	0.172561E-00	0.193252E-00	0.401146E-04	0.567212E-02
0.30E 01	0.14E 01	0.402344E-03	0.135917E-01	0.176464E-00	0.196308E-00	0.709991E-04	0.266816E-02
0.30E 01	0.16E 01	0.265179E-03	0.633540E-02	0.178497E-00	0.197366E-00	0.473335E-04	0.125039E-02
0.30E 01	0.18E 01	0.144065E-03	0.302217E-02	0.179171E-00	0.197033E-00	0.258123E-04	0.595468E-03
0.30E 01	0.20E 01	0.733293E-04	0.148733E-02	0.178861E-00	0.195741E-00	0.131157E-04	0.291131E-03
0.30E 01	0.22E 01	0.365234E-04	0.757250E-03	0.177843E-00	0.193794E-00	0.649542E-05	0.146750E-03
0.30E 01	0.24E 01	0.180623E-04	[illegible]	0.176326E-00	0.191409E-00	0.318475E-05	0.763239E-04
0.30E 01	0.24E 01	[illegible]	[illegible]	0.174646E-00	0.188743E-00	0.154557E-05	0.409118E-04
0.30E 01	[illegible]	0.424898E-05	0.121328E-03	0.172328E-00	0.185904E-00	0.732210E-06	0.229556E-04
0.30E 01	0.30E 01	0.192492E-05	0.697357E-04	0.170055E-00	0.182977E-00	0.327339E-06	0.127600E-04
0.30E 01	0.32E 01	0.755147E-06	0.410487E-04	0.167683E-00	0.180012E-00	0.126625E-06	0.738927E-05
0.30E 01	0.34E 01	0.176440E-06	0.246854E-04	0.165261E-00	0.177053E-00	0.291587E-07	0.437063E-05
0.30E 01	0.36E 01	-0.966151E-07	0.151339E-04	0.162823E-00	0.174127E-00	-0.157312E-07	0.263522E-05
0.30E 01	0.38E 01	-0.211551E-06	0.944126E-05	0.160394E-00	0.171253E-00	-0.339315E-07	0.161684E-05
0.30E 01	0.40E 01	-0.245901E-06	0.598401E-05	0.157990E-00	0.168444E-00	-0.388499E-07	0.100792E-05
0.50E 01	0.20E-00	0.111836E-00	0.124910E 01	0.732226E-01	0.991569E-01	0.818895E-02	0.127826E-00
0.50E 01	0.40E-00	0.122225E-01	0.381822E-00	0.101821E-00	0.137015E-00	0.124451E-02	0.523154E-01
0.50E 01	0.60E 00	-0.107776E-01	0.189839E-00	0.121531E-00	0.162012E-00	-0.130981E-02	0.307561E-01
0.50E 01	0.80E 00	-0.532017E-02	0.101161E-00	0.135873E-00	0.179117E-00	-0.722866E-03	0.181198E-01
0.50E 01	1.00E 00	-0.103134E-02	0.503508E-01	0.146445E-00	0.190760E-00	-0.151034E-03	0.960492E-02
0.50E 01	0.12E 01	0.152034E-03	0.236302E-01	0.154235E-00	0.198509E-00	0.234690E-04	0.469079E-02
[illegible]	[illegible]	0.255491E-03	0.105532E-01	0.159928E-00	0.203455E-00	0.408922E-04	0.220813E-02
0.50E 01	0.16E 01	0.164783E-03	0.502095E-02	0.164017E-00	0.206372E-00	0.270271E-04	0.103618E-02
0.50E 01	0.18E 01	0.871528E-04	0.235009E-02	0.166865E-00	0.207813E-00	0.145427E-04	0.494614E-03
[illegible]	[illegible]	[illegible]	0.116613E-02	0.168747E-00	0.208175E-00	0.724047E-05	0.242763E-03
0.50E 01	0.22E 01	0.205219E-04	0.592819E-03	[illegible]	[illegible]	0.348606E-05	0.123114E-03
0.50E 01	0.24E 01	0.968675E-05	0.312503E-03	0.170396E-00	0.206737E-00	0.165058E-05	[illegible]
[illegible]	0.24E 01	0.452554E-05	0.170792E-03	0.170449E-00	0.205503E-00	0.771373E-06	0.350641E-04
0.50E 01	0.28E 01	0.207969E-05	0.945456E-04	0.170127E-00	0.203551E-00	0.353813E-06	[illegible]
0.50E 01	0.30E 01	0.924011E-06	0.563118E-04	0.169508E-00	0.201598E-00	0.156827E-06	0.113523E-04
0.50E 01	0.32E 01	0.381446E-06	0.338043E-04	0.168654E-00	0.199480E-00	0.643249E-07	0.675526E-05
0.50E 01	0.34E 01	0.130724E-06	0.208340E-04	0.167616E-00	0.197256E-00	0.219116E-07	[illegible]
0.50E 01	0.36E 01	0.167227E-07	0.131514E-04	0.166433E-00	0.194966E-00	0.311507E-08	[illegible]
0.50E 01	0.38E 01	-0.278019E-07	0.844419E-05	0.165138E-00	0.192638E-00	[illegible]	[illegible]
0.50E 01	0.40E 01	-0.439887E-07	0.553216E-05	0.163758E-00	0.190296E-00	[illegible]	[illegible]

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